Duality In Lattice Gauge Theory

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by

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This thesis is dedicated to individuals who aspired to pursue higher education or research but had to give up their dreams due to various reasons.

Abstract

In this thesis, we have constructed exact duality in pure SU(N) Hamiltonian lattice gauge theory in the (2+1) dimension. This thesis work is based on a general and systematic canonical transformation method to obtain duality and local dual dynamics in all SU(N) lattice gauge theories. This method treats spin, Abelian, and non-Abelian dualities on the same footing. A set of iterative canonical transformations over the entire lattice is used to transform the original SU(N) Kogut-Susskind electric description into the magnetic description by defining plaquette holonomies over every plaquette as the new fundamental variables. The resulting dual description is in terms of the SU(N) magnetic scalar fields or plaquette flux loops and their conjugate electric scalar potentials. Under SU(N) gauge transformations they both transform like adjoint matter fields. The dual Hamiltonian describes the nonlocal self-interactions of these plaquette flux loops in terms of their electric scalar potentials and with inverted coupling. We show that these nonlocal loop interactions can be made local and converted into minimal couplings by introducing SU(N) auxiliary gauge fields along with new plaquette constraints.

Exploiting these exact duality transformations, we construct the most general disorder operator for SU(N) lattice gauge theory. These disorder operators, defined on the plaquettes and characterized by (N-1) angles, are the creation and annihilation or the shift operators for the SU(N) magnetic vortices carrying (N-1) types of magnetic fluxes. They are dual to the SU(N) Wilson loop order operators which, on the other hand, are the creation-annihilation or shift operators for the (N-1) electric fluxes on their loops. We derive a new order-disorder algebra involving SU(N) Wigner-D matrices that reduce to standard Wilson-'t Hooft algebra in a special limit. We also compute the path integral expression for the vortex-free energy, which should be useful for Monte-Carlo simulations and to understand the role of magnetic vortices and their condensation, if any, in the colour confinement.

Furthermore, we construct an SU(N) toric code model describing the dynamics of SU(N) electric and magnetic fluxes on a two-dimensional torus. We show that the model has N^2 topological degenerate ground states, which have been constructed in terms of coherent superposition of all possible spin network states on the torus with Wigner coefficients as their amplitudes. The electric and magnetic field excitations are constructed using SU(N) link holonomies and the SU(N) magnetic vortex creation operator respectively. We also show that the new SU(N) order-disorder algebra leads to the non-Abelian anyonic nature of these excitations or quasi-particles and their statistics is encoded in the Wigner-D functions. Having non-Abelian anyons, this model has the potential applications in topological quantum computing.

Publications

The thesis is based on the following publications:

I Manu Mathur, Atul Rathor, "**Exact duality and local dynamics in SU(N) lattice gauge theory**", Physical Review D **107**, 074504(2023)

II Manu Mathur, Atul Rathor, "**Disorder operators and magnetic vortices in SU(N) Lattice Gauge Theory**", arXiv:2307.06278 (2023)

III Manu Mathur, Atul Rathor, "**SU(N) toric code and non-Abelian anyons**", Phys. Rev. A **105**, 052423 (2022)

my other publications are:

IV Manu Mathur, Atul Rathor, "**Spin Networks, Wilson Loops and 3nj Wigner Identities**", arXiv:2111.14685 (2021)

V Manu Mathur, Atul Rathor, T. P. Sreeraj, "**Resolution of SU(3) outer multiplicity problem and the SU(3) $\otimes$ SU(3) invariant group SO(4,2)**", arXiv:1906.10410 (2019)

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Regards,

Atul Rathor,
Senior Research Fellow
S N Bose National Center for Basic Science-Kolkata

CONTENTS

CHAPTER 1

INTRODUCTION

Gauge theories play a fundamental role in our understanding of nature. They are based on local gauge symmetries at each point in space-time under a gauge group [1]. All fundamental forces of nature are described by gauge theories. In quantum gauge theories, local gauge invariance induces interaction between matter particles through its excitations called "gauge bosons". The most prominent example of an Abelian gauge theory is the theory of photons interacting with electrons and positrons called quantum electrodynamics (QED). QED has been very successful in explaining the nature of electromagnetic forces with very high accuracy. Other examples of gauge theories with larger symmetry groups are Yang-Mills theories which are based on non-Abelian special unitary gauge group SU(N). This larger symmetry group allow us to assign an additional label to both gauge and matter fields known as "colour charge" or simply "colour" which is the non-Abelian analogue of electric charge. Therefore, the theory of strong interactions between quarks and gluons with SU(3) symmetry is called quantum chromodynamics (QCD). QCD is an asymptotically free theory. This means that at high energy the coupling of the theory is small. This high-energy regime can be handled in terms of the perturbation theory. On the other hand, at low energies or long distances, the coupling becomes strong and the perturbation theory breaks down. We thus need new approaches to handle the low energy regime. In particular, the observation that isolated quarks do not exist or equivalently the problem of colour confinement is one of the most fundamental problems in QCD which requires new nonperturbative approaches to analyze gauge theories.

In 1974, Wilson proposed a non-perturbative method of regularization of aforementioned theories by discretising space and time [2–6]. Lattice spacing 'a' acts as a regulator of the theory and provides a natural cutoff to momentum and in turn, ultraviolet infinities disappear. Wilson's formulation is based on the Euclidean path integral which converts a gauge theory into a statistical mechanical model whose temperature is mapped to the inverse of the coupling of the gauge theory. This formulation enables one to use the standard techniques of statistical physics, such as mean-field and variational methods, strong coupling expansion (high-temperature expansion) and Monte Carlo (MC) simulations [4]. Once lattice computations are done, one has to take the continuum limit which corresponds to taking the lattice spacing to zero. As the coupling parameter of the theory will be 'a' dependent, one has to renormalize the coupling before setting 'a' to zero.

The continuum theory can easily be recovered from their lattice version by tuning a set of relevant parameters. The quantum field theory which describes nature can be defined as the limit of a regulated theory with a short-distance ultraviolet cutoff 'a' and the volume L.

$$\text{Quantum Field Theory} = \lim_{a\to 0,\, L\to\infty} (\text{Lattice Field Theory})_{a,\, L}$$

All the quantities in the lattice formulation such as lattice mass m_L and correlation length ξ_L are measured in terms of a number of lattice spacing and therefore unit-less. Their corresponding physical quantities can be obtained by absorbing the appropriate power of lattice spacing and followed by the continuum limit.

$$m_{\text{phys}} = \lim_{a\to 0,\, L\to\infty} \frac{m_L}{a}, \qquad \xi_{\text{phys}} = \lim_{a\to 0,\, L\to\infty} a \times \xi_L \tag{1.1}$$

As the lattice constant approaches zero, the lattice correlation must diverge in order to obtain a finite physical correlation length. In terms of statistical field theory, this divergence of the correlation length corresponds to a phase transition. To obtain a finite physical mass, the lattice mass should depend on the lattice spacing. In fact, any lattice quantity is a function of the coupling parameter of the lattice theory, which also varies with the lattice spacing a. To obtain the continuum limit, the lattice coupling must be determined as a function of the lattice spacing, and then the RG (renormalization group) fixed point must be found at which the coupling does not vary with the lattice cutoff. The dependence of the coupling parameter on lattice spacing is defined through the $\beta(g)$ function.

$$\beta(g) = a\frac{d}{da}g(a) \tag{1.2}$$

Renormalization calculations in (3+1) dimensions of Yang-Mills theory provide the following beta function:

$$\beta(g) = \beta_0 g^3 + \beta_1 g^5 + O(g^7) \tag{1.3}$$

Where $\beta_0 = \frac{11}{3}\left(\frac{\text{N}}{16\pi^2}\right)$ and $\beta_1 = \frac{34}{3}\left(\frac{\text{N}}{16\pi^2}\right)^2$. One can integrate it to get lattice spacing as

$$a = \frac{1}{\Lambda_{\text{lattice}}}(\beta_0 g^2)^{\frac{\beta_1}{2\beta_0^2}} e^{-\frac{1}{2\beta_0 g^2}} \left(1 + O(g^2)\right) \tag{1.4}$$

In the above equation, Λ_{lattice} is the constant of integration. It is clear that the continuum limit occurs near $g^2 \to 0$. In $(2+1)$ dimension the physical coupling constant e^2 has dimension of mass and the dimensionless coupling is defined as $g^2 \equiv ae^2$. All physical quantities scale according to their mass dimensions:

$$m_L = m_{\text{phys}}\, a \approx cg^2. \tag{1.5}$$

The Hamiltonian formulation by Kogut-Susskind discretizes space and keeps time continuous. The gauge symmetries are imposed through local Gauss law constraints. The Hamiltonian formulation is intuitively more appealing as it directly deals with fundamental operators of the theory and its Hilbert space. The question regarding mass gap, glueball and hadron spectrum can be directly posed. In principle Hamiltonian approach allows us to diagonalize the Hamiltonian nonperturbatively, but in reality, it is very difficult due to infinite dimensional Hilbert space and complex dynamics. Various truncation schemes are applied for computational purposes. The oldest and perhaps simplest scheme is strong coupling approximations $g^2 \to \infty$ which allows us to calculate the low energy spectrum but these results are completely unphysical as the continuum limit lies near $g^2 \to 0$. Various variational methods with the trial wave functions are used but their success is limited by the right choice of variational ansatz. Other methods includes t-expansions [7, 8], plaquette expansion [9] and coupled cluster method [10–12]. In the past decade, new technology development as tensor-network and the beginning of quantum computations in gauge theory has renewed the interest in the Hamiltonian formulations.

To date, a lot of distinct features and phenomena have been discovered in lattice gauge theory. Lattice gauge theory is now an interdisciplinary tool. They describe an incredibly wide range of phenomena from high energy physics, the fundamental interactions of the standard model [13], and high-temperature quark-gluon plasma [14,15] to topologically ordered materials in condensed matter physics [16, 17]. Most advanced and powerful tools like matrix product state tensor networks are being applied for lattice gauge theory [18]. In the past few years, there has been an immense effort to implement lattice gauge theory in artificially engineered quantum simulators [19]. The most crucial issue in the quantum simulations of gauge theories is the implementation of Gauss' law or local symmetries which are not physical symmetries but obeyed by the fields of the theory. [20–23]. Recently, a few experiments have successfully implemented these symmetries for simpler lattice gauge theories in (1+1) dimension, such as the Schwinger model, using trapped ions [24, 25] and ultracold atoms [22, 26]. There are proposals to extend these schemes to more complex theories and in higher dimensions. Treatment of non-Abelian gauge theory with ultracold atom, representation of infinite dimensional link Hilbert space, still remains a very challenging problem [27]. In the last few years, there have been many efforts to search for various representations of gauge theories which can be more suitable for quantum simulations. In the case of Abelian gauge theories, it has been claimed that the dual magnetic field description will be more cost-effective for quantum computations [28] as compared to the electric description. In the recent past there has been a surge in the search for dualities of lattice gauge theory and their application to quantum computations [20, 28]

This thesis focuses on exact duality transformations in pure SU(N) Hamiltonian lattice gauge theory in the (2+1) dimension. The resulting dual description (see chapter 5) is in terms of the magnetic scalar fields or plaquette loops & their conjugate electric scalar potentials. Under SU(N) gauge transformations they both transform like adjoint matter fields. The dual Hamiltonian describes the nonlocal interactions of the plaquette loops with inverted coupling in terms of the electric scalar potentials. We show that these nonlocal loop interactions can be made local and converted into minimal couplings by introducing SU(N) auxiliary gauge fields along with new plaquette constraints. We briefly discuss the concepts of duality, disorder operators in spin systems and lattice gauge theories as they are extensively studied in the latter part of the thesis.

1.1 Duality

Dualization of the models is a well-established concept in statistical mechanics and quantum field theory in the continuum as well as lattice formulations [29–34]. Duality refers to an exact relationship between two descriptions of the same theory. More precisely, duality is a bijective map between two well-defined classical or quantum theories. Duality in the statistical model relates one theory at low temperature to another theory at high temperature and vice versa. In quantum field theory, duality transformation maps the strong coupling regime of the theory to the weak coupling regime of the dual model. Thus, duality makes it possible to investigate the strongly coupled region of a theory by a perturbative expansion of its dual theory.

There are many dualities known in physics with different characteristics and natures [29]. The oldest known duality is electromagnetic duality in Maxwell's theory. There is an evident similarity in the role of electric and magnetic fields in Maxwell's equations in the absence of sources first noticed by Heaviside in 1884. This duality is later extended to the quantum domain by Paul Dirac in his seminal work "The theory of magnetic poles" [35]. In 1934 Kramers and Wannier discovered self-duality in the statistical mechanical system of two dimensional Ising model which elegantly relates a high-temperature classical Ising model on a square lattice to the same model at low-temperature [36]. This self-duality was used to predict the existence of ferromagnetic phase transition even before its solution was discovered. In the 2-dimensional XY model, which undergoes a defect-driven phase transition, the duality transformations make these topological defect degrees of freedom manifest [37, 38]. The earliest example of duality in lattice gauge theory is gauge-spin duality by Franz Wegner in 1971 [37]. He showed that a (2+1) dimensional Z_2 lattice gauge theory can be exactly mapped into a (2+1)-d Z_2 Ising model describing Ising magnets. Another feature of duality transformations is that they provide an alternative and simple way to look at features like disorder operators [30, 37, 39] and topological charges [40, 41] which are extremely hard to investigate in the original formulation. As a consequence of the above properties, the dual formulations of Abelian lattice gauge theories have led to interesting confining and non-confining phases in terms of topological magnetic monopoles [38, 40]. The other well-known example of duality is the equivalence between the Sine-Gordan and the massive Thirring model in $(1 + 1)$ dimension where the solitonic degrees of freedom of the former get identified with the Fermionic degrees of freedom of the latter [42]. Duality transformations also play an important role in supersymmetric Yang-Mills theories and string theories [43–46]. In this thesis we will only focus on dualities in SU(N) lattice gauge theories and their connections to spin models.

In the past few decades, there have been a number of approaches to reformulate SU(N) Yang-Mills theories directly in terms of the magnetic or equivalently plaquette loop variables on lattice. All these approaches attempt to solve the non-Abelian Gauss laws [47–68] to define dual electric potentials which are conjugate to the corresponding SU(N) magnetic fields. The various solutions of the non-Abelian Gauss laws or the duality relations are involved [29, 38, 48, 49, 67, 69–77] and difficult to interpret. As a result, the dynamics in terms of the dual electric potentials get complicated [49,69] and is often nonlocal [78,79]. These involved/nonlocal interactions also make the dual description computationally unwieldy. Further, establishing the equivalence between the Hilbert spaces before and after duality is not easy. Many of these duality approaches are specific to SU(2) gauge group and the generalization to SU(N) group is far from clear. As an example, in one of the earliest approaches [70], a polar decomposition of the covariant electric fields is used to solve the SU(2) Gauss law constraints. However, the resulting magnetic field term in the SU(2) gauge theory Hamiltonian is technically involved and difficult to work with. Also such a polar decomposition for SU(3) or higher SU(N) gauge group is not clear. Some of the earlier approaches to non-Abelian duality or loop formulation leading to gauge invariant operators are motivated by gravity [72, 80–85]. A gauge invariant metric or dreilbein tensor is constructed out of the covariant SU(2) electric or magnetic field. The problem with such approaches is the exact equivalence between the initial and final (gauge invariant) coordinates is not simple. In Nair-Karabali [86–88] approach the SU(N) vector potentials enable us to define gauge covariant matrices leading to gauge invariant coordinates which are then quantized to analyze the theory directly in the physical Hilbert space $\mathcal{H}^{\text{phys}}$. Formulations of the lattice gauge theory in terms of gauge invariant Wilson loops were among the earliest attempts at dualization. The motivation was to go from link fields which create coloured excitation to loops which could create gauge invariants excitations. However, these loop formulations suffer the problem of nonlocality and Mandelstam constraints. As there exist a huge number of Wilson loops with different shapes and sizes so these loops are over-complete and satisfy some constraints known as Mandelstam constraints. A systematic method for transitioning from the standard Kogut-Susskind link formulation of pure SU(N) gauge theory to a loop formulation using fundamental loop operators and loop states is still missing in the literature. [52, 71, 73, 89].

In this thesis, using the Hamiltonian approach in (2+1) dimensions, we illustrate how to evade the above difficulties and transit from the original SU(N) Kogut-Susskind (K-S) electric vector field & magnetic vector potential description (see 3.20) to the magnetic scalar field & electric scalar potential dual description (see chapter 5). The thesis work is based on a general and systematic canonical transformation method to obtain duality and local dual dynamics in all SU(N) lattice gauge theories [78, 79, 90, 91]. A set of iterative canonical transformations over the entire lattice is used to transform the original Kogut-Susskind electric description [56] into the magnetic description by defining fields over every plaquette as the new fundamental variables. The canonical transformations ensure that the degrees of freedom remain intact and the dual magnetic description is exactly equivalent to the original electric description. We make the resulting dual dynamics local and manifestly covariant by introducing new auxiliary gauge fields with local plaquette constraints. The resulting non-Abelian duality transformations can be interpreted as a simple SU(N) generalization of the U(1) duality transformations. As expected, the duality transformations interchange the interacting and the non-interacting parts of the Hamiltonian and the gauge coupling gets inverted. The above method when applied to the simpler Ising model or Z_2 gauge theory leads to the well known Kramers-Wannier and Wegner dualities respectively [78]. Thus the method treats spin, Abelian and non-Abelian dualities on the same footing.

In our previous work [78] we have constructed duality transformations through a series of canonical transformations which explicitly solved the SU(N) Gauss law constraints at every lattice site. The dual theory in this case was a SU(N) spin model without any gauge degrees of freedom. However, the above solutions of SU(N) Gauss law constraints are nonlocal relations between the SU(N) Kogut-Susskind electric fields and the dual electric scalar potentials. This nonlocality led to nonlocal dual Hamiltonian [78, 79]. The above duality and these nonlocality issues have been discussed recently in the context of quantum simulations in the magnetic basis (see Bauer et. al. in [92].) In the present work, we take a different route and define the dual SU(N) electric scalar potentials without solving the Gauss law constraints. In this approach, we define SU(N) magnetic scalar from local plaquette loop holonomies & their conjugate electric scalar potentials are obtained using canonical transformations.

1.2 Disorder operator

Disorder operators, introduced originally in 1971 by Kadanoff and Ceva in the context of the two-dimensional Ising model [39], have been widely discussed and found useful in the studies of phase structures of spin models as well as Abelian and non-Abelian gauge theories. They also play a pivotal role in differentiating topological phases of matter [37] and in Boson-Fermion transmutation through 'order $\times$ disorder' combinations. It is generally known that the duality transformations in spin models and gauge theories naturally lead to these disorder operators as the fundamental operators describing the dual interactions. The Kramers-Wannier duality in $(1+1)$ dimensional Ising spin model [36] and the Wegner duality in $(2+1)$ dimensional Z_2 gauge theory are the simplest examples which illustrate the above facts [3, 37, 93, 94]. In the $(1+1)$ dimensional Ising model the disorder operators are simply the dual spin operators which describe the dual interactions with inverse coupling. They also create Z_2 kinks which are responsible for disordering the ground state leading to the loss of magnetization above the Curie temperature.

In Abelian and non-Abelian gauge theories, the disorder operators acquire additional meaning and significance as the underlying duality transformations also interchange the roles of electric and magnetic degrees of freedom. The order (disorder) operators are simply the potentials (dual potentials) which are conjugate to electric (magnetic) fields respectively. They can therefore be interpreted as "translation operators" for electric and magnetic fluxes respectively. In 1978 't Hooft emphasized the role of disorder operators in the context of quark confinement in SU(N) gauge theory [95]. The 't Hooft disorder operator creates topological charges or magnetic fluxes which belong to the center Z_{N} of the gauge group SU(N). It is well known that the Wilson loop $W(C)$ is the natural order parameter for SU(N) lattice gauge theories which distinguishes the confining from the non-confining phase of a pure gauge theory [3]. A simple relation between these two operators which in now known as Wilson- 't Hooft algebra indicates the dual nature between them. However, it has been noticed in the past that the 't Hooft operator which is characterized by the centre of the gauge group is not the most general disorder operator as it create only Z_{N} magnetic fluxes [96, 97]. The main difficulty in constructing these nonlocal operators is their nonlocal nature. Therefore without knowing the exact duality relation, they are extremely hard to guess. However, in the dual model, these are fundamental local fields. The deep connection between duality and disorder operators has also been extensively discussed in the past [3, 29, 98]. Exploiting exact duality transformations discussed in the present work we construct the most general disorder operator for SU(N) lattice gauge theory which creates SU(N) magnetic fluxes.

1.3 Duality, anyons and quantum computations

Nearly five decades ago, Feynman proposed the idea of "Simulating physics with computers " [99]. He suggested that quantum systems should be studied using other quantum systems. This idea was further developed as the concept of a "Universal quantum simulator" which in principle can simulate any quantum system that evolves according to local interactions [100]. Despite significant progress in the field of quantum computing, the fabrication of quantum computer hardware remains highly challenging. Existing candidates for quantum computers are quantum systems existing at single atom scale. One of the main challenges in building these quantum computers has been to protect their qubits from environmental interaction or decoherence. One possible approach to overcome the decoherence problem is to store the quantum information in anyonic states with topological charges so that they are naturally protected against any local perturbation. Therefore in the last few years topological quantum computing (TQC) with Abelian & non-Abelian anyons has become one of the most promising approaches for storing and processing quantum information reliably in systems on two-dimensional surfaces [101–106]. As anyonic states carry both electric & magnetic charges, we need order as well as disorder operators to construct them explicitly (see Chapter 7).

In the above quantum computational approaches, the quantum information is encoded in the topological braiding of anyons making it resilient against any local perturbations or decoherence effects. All these models with anyons originated from the simple Z_2 toric code model proposed by Alexei Kitaev in 1997 [101]. This $(2+1)$ dimensional model is an exactly solvable model with 4 degenerate ground states which are characterized by $Z_2 \otimes Z_2$ topological order. The anyonic excitations in this model and the topological nature of their braiding have led to the idea of topological or fault-tolerant quantum computers having an intrinsic resistance to small perturbations. In the past, the appealing property of topological ordering and quantum error correction of the Z_2 toric code model has led to other interesting models with discrete Abelian and non-Abelian groups [107–109]. Kitaev himself proposed a model of quantum doubles [101]. Levin and Wen introduced their string net models with string net condensation as the basic mechanism underlying the topological ordering [110].

In this thesis, we have generalized Kitaev's Z_2 toric code model to SU(N) group leading to non-Abelian anyons. In the recent past models with non-Abelian anyons have been subject of intense research for their quantum computing applications [102, 103]. The SU(N) toric code model, discussed in this work, provides a natural setting for such non-Abelian exotic quasiparticles as electric and magnetic excitations over the ground states. This SU(N) model has N^2 degenerate ground states which are loop or spin network states and are characterized by $Z_\mathrm{N} \otimes Z_\mathrm{N}$ topological charges. They are explicitly constructed in terms of spin networks and the Wigner coefficients as their amplitudes. In fact, besides their topological stability, the ground states of SU(N) toric code model are also geometrically rigid because of the numerous interlinked triangular electric flux constraints satisfied by the underlying spin network states over the entire torus. This geometrical stability is purely a group theoretical property of the Hilbert space of the SU(N) toric code model. The spin networks, constructed in section 7.1.1, are similar to the string nets of Levin and Wen which were discussed in the context of topological phases and non-Abelian anyons [110]. Both string nets and spin networks represent the physical Hilbert spaces after solving the Gauss law constraints. We also construct all excited states using SU(N) link holonomies and non-Abelian vortex creation-annihilation operator. We show that the SU(N) canonical commutation relations between electric fields and the conjugate potentials lead to the non-Abelian anyonic nature of these excitations or quasiparticles. We further show that their mutual non-Abelian statistics is encoded in Wigner D matrices.

Overview of thesis

The primary aim of this thesis is to develop exact duality transformations for non-Abelian lattice gauge theories and to use these transformations to construct the most general disorder operators in SU(N) lattice gauge theory. Further, we generalize Z_2 toric code model to SU(N) group and use the above SU(N) disorder operator to create excitations which are non-Abelian anyons.

The brief outline of the thesis is the following:

In chapter (2), we review known duality in the spin model and Abelian lattice gauge theory by means of canonical transformations. Using canonical transformations, we construct the Kramers-Wanneir duality in (1+1) dimensional Ising model, Wegner duality in $2+1$ Z_2 lattice gauge theory and electromagnetic duality in (2+1) compact U(1) lattice gauge theory. We also construct and discuss disorder operators in these systems.

In Chapter (3), we introduce the Kogut-Susskind Hamiltonian formulation of lattice gauge theory. We briefly discuss the problem of Mandelstam constraints in the loop formulation of lattice gauge theory.

In chapter (4) we generalize Wegner Z_2 gauge-spin duality to SU(N) lattice gauge theory in (2+1) dimension. We solve non-Abelian Gauss law through a series of iterative canonical transformations to obtain a dual SU(N) spin model which has global SU(N) gauge invariance in contrast to local gauge symmetries in the original theory.

In chapter (5), we construct exact duality transformations for (2+1) dimension SU(N) lattice gauge theory on a finite lattice. The dual theory is described in terms of magnetic field and dual vector potentials. The dual theory has only local interaction in terms of the dual potentials. As original Kogut-Susskind formulation contained gluon-gluon interaction of all orders and therefore suffer from computational difficulties.

In Chapter (6), we introduce and construct disorder operator for SU(N) lattice gauge theory. There we show that the plaquette disorder operator creates a magnetic vortex on the plaquette or the dual site. These disorder operators also satisfy generalized Wilson-'t Hooft algebra which reduces to known Wilson-'t Hooft algebra in a special case. The partition function representation and the free energies of these SU(N) magnetic vortices are discussed.

In Chapter (7), we generalize the Kitaev toric code model to an SU(N) group. This model is also exactly solvable and contains non-Abelian anyons which are useful for topological quantum computations. We show that these excitations follow non-Abelian anyonic statistics.

In Chapter (8), we summarize our work briefly. All the technical details are worked out in the Appendices.

CHAPTER 2

SPIN SYSTEMS

In this chapter, we review the well known dualities in the Z_2 spin or Ising model, Z_2 lattice gauge theory and then in pure Abelian lattice gauge theories. The above duality relations are relatively simple and can be obtained using various textbook methods. However, our purpose to revisit them is to obtain them using the new canonical transformation method discussed in this thesis. This exercise also gives us confidence in using this new method to explore dualities in non-Abelian gauge theories discussed in Chapters 4 and Chapter 5. In this chapter using canonical transformations, we will first establish the Kramers-Wanneir duality in (1+1)-d Z_2 Ising model followed by Wegner duality in (2+1)-d Z_2 gauge theory and finally U(1) or electromagnetic duality in (2+1)-d compact U(1) lattice gauge theory. As mentioned earlier, this chapter will set the stage for the subsequent chapters on the non-Abelian SU(N) duality transformations. We will show that the non-Abelian duality transformations obtained by the canonical transformation method are simple and straightforward generalization of Abelian duality transformations (compare (2.3) in U(1) case and (5.1.2) in SU(N) case). To the best of our knowledge such correspondence between Abelian and non-Abelian duality transformations have not been discussed in the literature.

In this chapter, we will also discuss the deep relationship between duality and disorder operators in these simpler spin models. They follow interesting order-disorder algebra. This will again help us understand the most general disorder operator in SU(N) lattice gauge theories constructed in Chapter 6.

The plan of this chapter is as follows: In section 2.1 we introduce (1+1)-d quantum Ising model and construct its dual model through a series of explicit canonical transformations. We also construct the Ising disorder operator and the corresponding order-disorder algebra. In section 2.2, we introduce Z_2 gauge theory which is described in terms of Z_2 gauge field defined on links. Again using a series of iterative canonical transformations, we convert them into Z_2 spin model where spins are defined on the plaquettes or on dual lattice sites. We also construct Z_2 disorder operators and establish the corresponding order-disorder Wilson-'t Hooft algebra. In section (2.3), we introduce U(1) lattice gauge theory defined in terms of link operators (magnetic vector potential) and their conjugate electric field operators. Using the canonical transformation method we reformulate theory in terms of plaquette operators (magnetic fields) and their conjugate electric potentials. We also construct U(1) disorder operators and corresponding order-disorder algebra.

2.1 Kramers-Wannier duality

Kramers-Wannier duality was the first and the simplest duality, apart from electromagnetism, to be constructed. In this section, we apply canonical transformations to $(1+1)$ dimensional Ising model to get the Kramers-Wannier duality. The Ising Hamiltonian in one space dimension is in terms of the canonically conjugate operators $\{\sigma_1(m);\sigma_3(m)\}$ at every lattice site $m = 0, 1, 2, \cdots$ satisfying,

$$\sigma_1^2(m) = 1; \qquad \sigma_3^2(m) = 1; \qquad [\sigma_1(m), \sigma_3(m)]_+ = 0. \tag{2.1}$$

In (2.1) $[A, B]_+ = AB + BA$. The Hamiltonian is

$$H \;=\; \sum_{m=0}^{\infty} \Big[\sigma_1(m) - \lambda\; \sigma_3(m)\sigma_3(m+1)\Big]. \tag{2.2}$$

The Kramers-Wannier duality is obtained by the following iterative canonical transformations along a line with $\bar{\sigma}_3(m=0) \equiv \sigma_3(m=0)$ and $\bar{\sigma}_1(m=0) \equiv \sigma_1(m=0)$:

$$\begin{aligned} &\mu_1(m) \equiv \bar{\sigma}_3(m)\sigma_3(m+1), && \mu_3(m) = \bar{\sigma}_1(m) \\ &\bar{\sigma}_3(m+1) = \sigma_3(m+1), && \bar{\sigma}_1(m+1) = \bar{\sigma}_1(m)\sigma_1(m+1) = \mu_3(m)\sigma_1(m+1). \end{aligned} \tag{2.3}$$

The above canonical transformations iteratively replace the conjugate pair $\{\sigma_1(m);\sigma_3(m)\}$ or equivalently $\{\bar{\sigma}_1(m);\bar{\sigma}_3(m)\}$ by a new conjugate pair $\{\mu_1(m);\mu_3(m)\}$. These new pairs are mutually independent and also satisfy the canonical relations (2.1). Unlike gauge theories (to be discussed in the next section), there are no spurious (string) degrees of freedom. This process is graphically illustrated in Figure 2.1. The relations (2.3) lead to,

$$\mu_3(m) = \prod_{s=0}^{m} \sigma_1(s). \tag{2.4}$$

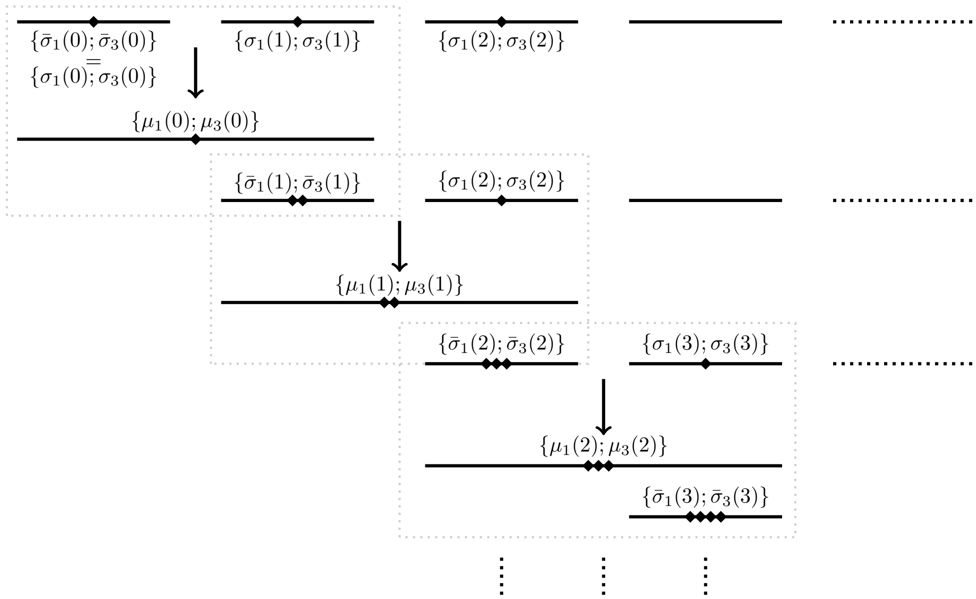


Figure 2.1: Kramers-Wannier duality through canonical transformations. The first three steps of duality or canonical transformations in (2.3) are illustrated.

The relations (2.4) can be easily inverted to give $\sigma_1(m) = \mu_3(m)\mu_3(m-1)$ with the convention $\mu_3(m=-1) \equiv 1$. The Ising model Hamiltonian can now be rewritten in its self-dual form in terms of the new dual conjugate pairs $\{\mu_1(m);\ \mu_3(m)\}$:

$$H \ = \ \sum_{m=0}^{\infty} \Big[\mu_3(m)\mu_3(m+1) - \lambda\ \mu_1(m)\Big]. \tag{2.5}$$

Therefore,

$$H(\sigma;\lambda) = \lambda^{-1} H(\mu;\lambda^{-1}).$$

This is the famous Kramers-Wannier self-duality. Since σ variables and μ variables satisfy the same algebra, the above expression shows that the high λ behaviour of the system is the same as the low λ behaviour. In particular, the energy eigenvalues at different values of coupling are related by $E(\lambda) = \lambda^{-1}E(\lambda^{-1})$. So, if we assume that there is a critical value λ_c of λ where there is a phase transition, the mass gap $G(\lambda)$ vanishes at λ_c. The above expression says that if the mass gap $G(\lambda)$ vanishes at λ_c it should also vanish at λ_c^{-1}. Therefore, if there is a single critical point then, $\lambda_c = 1$. This is a simple illustration of the usefulness of such dualities in the study of the phase structure of various systems. Another interesting and important feature of duality is that it has interchanged the interacting and non-interacting parts of the Hamiltonian on going from the $\{\sigma_1;\ \sigma_3\}$ to the dual $\{\mu_1;\ \mu_3\}$ variables. In other words, duality maps

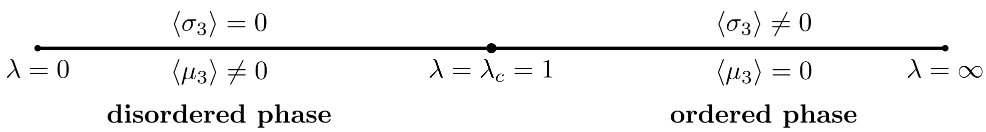


Figure 2.2: Duality and ordered & disordered phases of $(1+1)$ dimensional Ising model [3, 37, 40, 94, 98, 111–113].

strong-coupling region to the weak coupling region and vice versa.

2.1.1 Ising disorder operator

In Ising model the magnetization operator, $\sigma_3(m)$ is the order operator as its expectation value measures the degree of order of the σ_3 variables. It is zero for $\lambda < \lambda_c$ and non-zero for $\lambda > \lambda_c$. This implies that the $\lambda > \lambda_c$ phase spontaneously breaks the global Z_2 symmetry: $\sigma_3 \to -\sigma_3$. On the other hand, the dual Hamiltonian (2.2) implies that it is natural to define $\mu_3(m)$ as a disorder operator [3, 37, 40, 94, 98, 111–113]. The vacuum expectation value ${}_\lambda\langle 0|\mu_3(m)|0\rangle_\lambda$ is the disorder parameter. We also note that the disorder operator $\mu_3(x_0)$ acting on a completely ordered state (all $\sigma_3(m) = +1$ or -1), flips all σ_3 spins at $m < x_0$ and creates a kink at x_0. The resulting kink state is orthogonal to the original ordered state and the expectation value of the disorder operator μ_3 in an ordered state vanishes:

$$ {}_{\lambda=\infty}\langle 0|\mu_3(m)|0\rangle_{\lambda=\infty} = 0, \qquad {}_{\lambda=\infty}\langle 0|\sigma_3(m)|0\rangle_{\lambda=\infty} = 1. \tag{2.6}$$

On the other hand, at $\lambda = 0$, the dual description (2.2) implies that the Ising model is in ordered state with respect to μ_3. As a consequence, the disorder parameter does not vanish and order parameter vanishes:

$$ {}_{\lambda=0}\langle 0|\mu_3(m)|0\rangle_{\lambda=0} = 1, \qquad {}_{\lambda=0}\langle 0|\sigma_3(m)|0\rangle_{\lambda=0} = 0. \tag{2.7}$$

The relations (2.6), (2.7) are illustrated in Figure 2.2.

2.2 Wegner duality and Z_2 Spin Model

Z_2 lattice gauge theory is the simplest theory with gauge structure and many rich features. Due to their enormous simplicity compared to non-Abelian lattice gauge theories and the presence of a confining phase, they have been used as a simple theoretical laboratory to test various confinement ideas [40, 94, 98, 111–113]. They also provide an explicit realization of the Wilson-'t Hoofts algebra of order and disorder operators characterizing different possible phases of the SU(N) gauge theories [40, 94–96, 98, 111–117]. In 1964, Schultz, Mattis and Lieb showed that

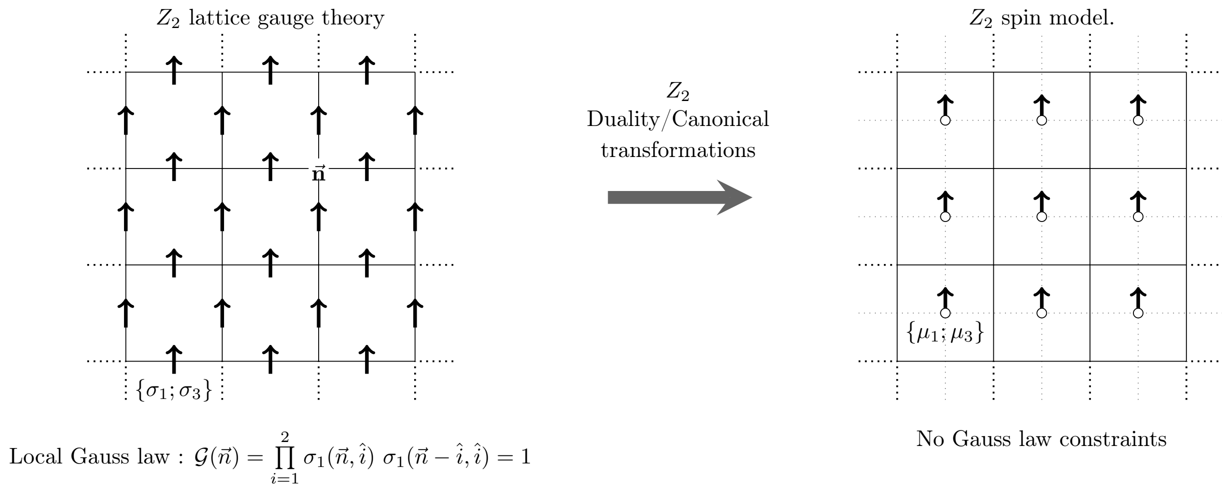


Figure 2.3: Duality between Z_2 lattice gauge theory and Z_2 (Ising) spin model. The initial and the final conjugate pairs $\{\sigma_1; \sigma_3\}$ and $\{\mu_1; \mu_3\}$, are defined on the links and the plaquettes or dual sites respectively. The corresponding SU(N) duality is illustrated in Figure 4.1.

the two-dimensional Z_2 Ising model is equivalent to a system of locally coupled fermions [118]. This result was later extended to Z_2 lattice gauge theory which also allows an equivalent description in terms of locally interacting fermions in (2+1) dimension [119,120]. These are old and well-known results. In the recent past, Z_2 (Z_{N}) lattice gauge theories have been useful to understand quantum spin models [16], quantum computations [101], tensor network or matrix product states [121] and their topological properties [122, 123], cold atom simulations [124]and entanglement entropy [125]. In view of these wide applications, the Z_2 (Z_{N}) lattice gauge theories and associated duality transformations are important in their own right. In 1971 F. Wegner [93, 126], using duality transformations, illustrated that in two spacial dimensions Z_2 lattice gauge theory can be mapped into a Z_2 Ising model which describes spin half magnets, see Figure 4.1. This is the earliest and the simplest example of the intriguing gauge-spin duality. This work by Wegner was strongly motivated by the self-duality of the planar Ising model discovered by Kramers and Wannier 30 years earlier [36, 127]. The two essential features of the Wegner duality [37] are

1. It eliminates all unphysical gauge degrees of freedom in Z_2 lattice gauge theory mapping it into Z_2 spin model with a Z_2 global symmetry. There are no Z_2 Gauss law constraints in the dual Z_2 spin model

2. It maps the interacting (non-interacting) terms in the Z_2 lattice gauge theory Hamiltonian into non-interacting (interacting) terms in the Z_2 spin model Hamiltonian resulting in the inversion of the coupling constant

It is important to note that the above gauge-spin duality is through the loop description of Z_2

lattice gauge theory. The original Hamiltonian is written in terms of fundamental Z_2 electric fields and their conjugate magnetic vector potentials. The Z_2 magnetic fields are not fundamental and obtained from magnetic vector potentials. On the other hand, in the dual Ising model the fundamental spin degrees of freedom are the Z_2 magnetic fields and their conjugate electric scalar potentials. Now the electric fields are not fundamental and are obtained from the electric scalar potentials. We arrive at this dual spin description through a series of canonical transformations. They convert the initial electric fields, magnetic vector potentials into the following two mutually independent *physical* & *unphysical* classes of operators:

1. *Z_2 spin or plaquette loop operators*: representing the *physical* Z_2 magnetic fields and their conjugate electric scalar potentials over the plaquettes (see Figure 2.5-a),

2. *Z_2 string operators*: representing the Z_2 electric fields and the Z_2 flux operators of the *unphysical* string degrees of freedom. These strings isolate all Z_2 gauge degrees of freedom (see Figure 2.5-b).

The interactions of spins in the first set are described by the Ising model. The corresponding physical Hilbert space is denoted by $\mathcal{H}^{\text{phys}}$. The second complementary set, containing Z_2 string operators, represents all possible redundant gauge degrees of freedom. We show that the Gauss law constraints freeze all strings leading to the Wegner gauge-spin duality within $\mathcal{H}^{\text{phys}}$. Further, the electric scalar potentials are shown to be the solutions of the Z_2 Gauss law constraints. *Note that no gauge fixing is required to obtain the dual description.*

The Z_2 lattice gauge theory involves Z_2 conjugate spin operators $\{\sigma_1(l); \sigma_3(l)\}$ on the link $l \in \Lambda$. The anti-commutation relations amongst these conjugate pairs on every link l are

$$\sigma_1(l)\ \sigma_3(l) + \sigma_3(l)\ \sigma_1(l) = 0. \tag{2.8}$$

They further satisfy: $(\sigma_3(l))^2 = (\sigma_1(l))^2 = 1$. In order to maintain a 1-1 correspondence with SU(N) lattice gauge theory (discussed in the chapter 4), it is convenient to identify the conjugate pairs $\{\sigma_1(l); \sigma_3(l)\}$ with Z_2 electric field, $E(l)$ and Z_2 vector potential, $A(l)$ as:

$$\sigma_1(l) = e^{i\pi E(l)}, \qquad \sigma_3(l) = e^{iA(l)}. \tag{2.9}$$

Above $E(l) \equiv \{0, 1\}$ and $A(l) \equiv \{0, \pi\}$. A basis of the two dimensional Hilbert space on each link l is chosen to be the eigenstates $|\pm, l\rangle$ of $\sigma_3(l)$ with eigenvalue ± 1 with $\sigma_1(l)$ acting as a spin flip operator:

$$\sigma_3(l)|\pm, l\rangle = \pm|\pm, l\rangle, \qquad \sigma_1(l)|\pm, l\rangle = |\mp, l\rangle. \tag{2.10}$$

The Z_2 lattice gauge theory Hamiltonian is given by

$$H = -\sum_{l\in\Lambda}\sigma_1(l) - \lambda\sum_{p\in\Lambda}\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4) \equiv H_E + \lambda H_B. \tag{2.11}$$

In (2.11) $\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4)$ represents the product of σ_3 operators along the four links of a plaquette. The sum over l and p in (2.11) are the sums over all links and plaquettes respectively. The parameter λ is the Z_2 gauge theory coupling constant. The first term H_E and the second term H_B in (2.11) represent the Z_2 electric and magnetic field operators respectively. The electric field operator $\sigma_1(l)$ is fundamental while the latter is a composite of the four Z_2 magnetic vector potential operators $\sigma_3(l)$ along a plaquette. After a series of canonical transformations, the above characterization of electric, magnetic field will be reversed. More explicitly, the dynamics will be described by the Hamiltonian (2.11) rewritten in terms of the fundamental magnetic field (the second term) and the electric field operator (the first term) will be composite of the dual electric scalar potentials (see (2.26a) and (2.31)).

The Hamiltonian (2.11) remains invariant if all 4 spins attached to the 4 links emanating from a site n are flipped simultaneously. This symmetry operation is implemented by the Gauss law operator $\mathcal{G}$:

$$\mathcal{G}(n) \equiv \prod_{l_n}\sigma_1(l_n) \tag{2.12}$$

at lattice site $n \in \Lambda$. In (2.12), $\prod_{l_n}$ represents the product over 4 links (denoted by l_n) which share the lattice site n in two space dimensions. The Z_2 gauge transformations are

$$\begin{aligned} \sigma_1(l) &\to \mathcal{G}^{-1}(n)\sigma_1(l)\mathcal{G}(n) = \sigma_1(l), \qquad \forall\ l \in \Lambda,\\ \sigma_3(l_n) &\to \mathcal{G}^{-1}(n)\sigma_3(l_n)\mathcal{G}(n) = -\sigma_3(l_n),\\ H &\to \mathcal{G}^{-1}(n)\ H\ \mathcal{G}(n) = H. \end{aligned} \tag{2.13}$$

Thus, under a gauge transformation at site n, the 4 link flux operator $\sigma_3(l_n)$ on the 4 links l_n sharing the lattice site n change sign. All other $\sigma_3(l)$ remain invariant. The physical Hilbert space $\mathcal{H}^{\text{phys}}$ consists of the states $|\text{phys}\rangle$ satisfying the Gauss law constraints:

$$\mathcal{G}(n)|\text{phys}\rangle = |\text{phys}\rangle \qquad \text{or} \qquad \mathcal{G}(n) \approx 1 \qquad \forall n \in \Lambda. \tag{2.14}$$

In other words, $\mathcal{G}(n)$ are unit operators within the physical Hilbert space $\mathcal{H}^{\text{phys}}$. All operator identities valid only within $\mathcal{H}^{\text{phys}}$ are expressed by $\approx$ sign. We now canonically transform this simplest Z_2 gauge theory with constraints (2.12) at every lattice site into Z_2 spin model without any constraints as shown in Figure 2.3. To keep the discussion simple, we start with a single plaquette ABCD shown in Fig. 2.4-a before dealing with the entire lattice. As the canonical transformations are iterative in nature, this simple example contains all the essential ingredients required to understand the finite lattice case. The four links AB, BC, CD, DA will be denoted by l_1, l_2, l_3, l_4 respectively. In this simplest case there are four Z_2 gauge transformation or equivalently Gauss law operators (2.12) at each of the four corners A, B, C and D:

$$\begin{aligned}
\mathcal{G}(A) &= \mathcal{G}(0,0) = \sigma_1(l_4)\sigma_1(l_1) \approx 1, \\
\mathcal{G}(B) &= \mathcal{G}(0,1) = \sigma_1(l_1)\sigma_1(l_2) \approx 1, \\
\mathcal{G}(C) &= \mathcal{G}(1,1) = \sigma_1(l_2)\sigma_1(l_3) \approx 1, \\
\mathcal{G}(D) &= \mathcal{G}(1,0) = \sigma_1(l_3)\sigma_1(l_4) \approx 1.
\end{aligned} \tag{2.15}$$

Note that these Gauss law operators satisfy a trivial operator identity:

$$\mathcal{G}(A)\ \mathcal{G}(B)\ \mathcal{G}(C)\ \mathcal{G}(D)\ \equiv\ 1. \tag{2.16}$$

The above identity states the obvious result that a simultaneous flippings at all 4 sites has no effect. This is because of the Abelian nature of the gauge group. We now start with the four initial conjugate pairs on links l_1, l_2, l_3 and l_4:

$$\begin{aligned}
&\{\sigma_1(l_1);\sigma_3(l_1)\}, \qquad && \{\sigma_1(l_2);\sigma_3(l_2)\}, \\
&\{\sigma_1(l_3);\sigma_3(l_3)\}, \qquad && \{\sigma_1(l_4);\sigma_3(l_4)\}.
\end{aligned} \tag{2.17}$$

Using canonical transformations we define four new but equivalent conjugate pairs. The first three string conjugate pairs:

$$\{\bar{\sigma}_1(l_1);\bar{\sigma}_3(l_1)\}, \quad \{\bar{\sigma}_1(l_3);\bar{\sigma}_3(l_3)\}, \quad \{\bar{\sigma}_1(l_4);\bar{\sigma}_3(l_4)\}$$

describe the collective excitations on the links AB, BC, CD and shown in Figures 2.4-a,b,c respectively. The remaining collective excitations over the plaquette or the loop $p \equiv ABCD$ are described by

$$\{\mu_1(p);\mu_3(p)\}$$

and shown in Figure 2.4-c. As a consequence of the three mutually independent Gauss law constraints $\mathcal{G}(B)$, $\mathcal{G}(C)$ and $\mathcal{G}(D)$, the three string electric fields are frozen to the value $+1$. Therefore there is no dynamics associated with the three strings. In other words, string degrees of freedom completely decouple from $\mathcal{H}^{\text{phys}}$. We are thus left with the final physical Z_2 spin operators $\{\mu_1(p);\mu_3(p)\}$ which are explicitly Z_2 gauge invariant. These duality transformations from gauge variant link operators to gauge invariant spin or loop operators are shown in Figure 2.3. To demonstrate the above results, we start with the initial link operators $\{\sigma_1(l_1);\sigma_3(l_1)\}$ and $\{\sigma_1(l_2);\sigma_3(l_2)\}$ as shown in Fig. (2.4)-a. As was done in $(1+1)$ dimensional Ising model,

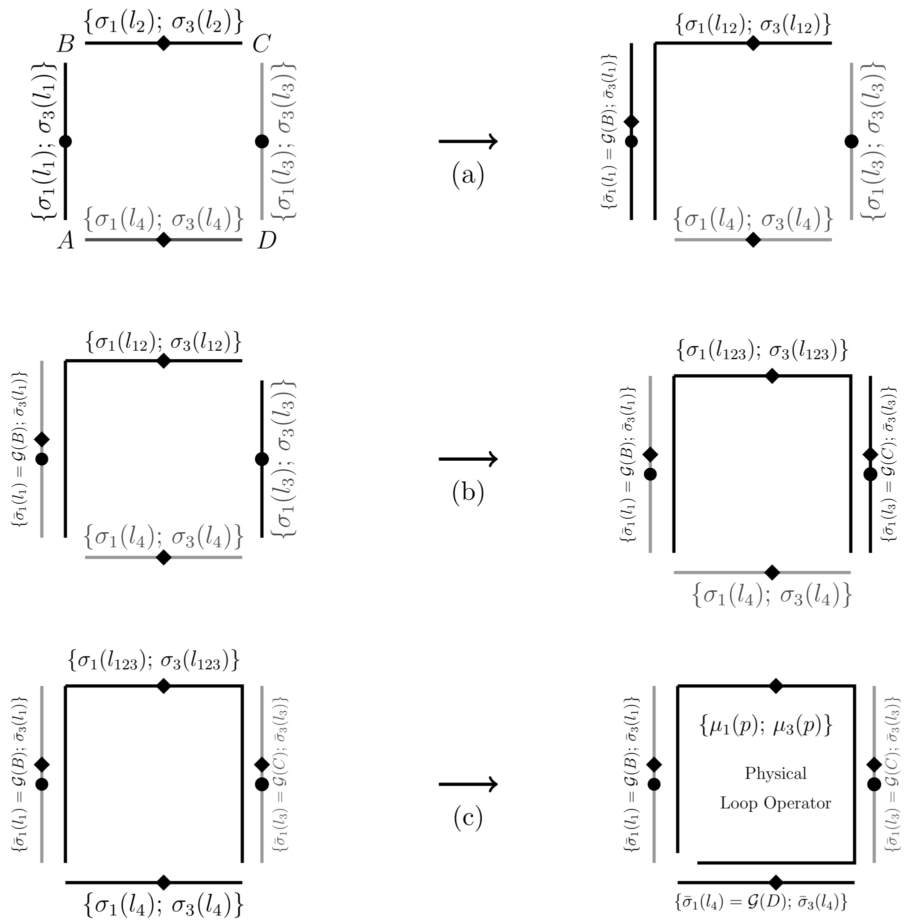


Figure 2.4: The Z_2 canonical transformations (2.18), (2.20), (2.21a) and (2.21b) are pictorially illustrated in (a), (b) and (c) respectively. The ♦ and • represent the electric fields of the initial horizontal and vertical links respectively.

we glue them using canonical transformations as follows:

$$\begin{aligned} \bar\sigma_3(l_1) &\equiv \sigma_3(l_1), & \sigma_3(l_{12}) &\equiv \sigma_3(l_1)\sigma_3(l_2) \\ \bar\sigma_1(l_1) &= \sigma_1(l_1)\sigma_1(l_2) \equiv \mathcal{G}(B), & \sigma_1(l_{12}) &= \sigma_1(l_2). \end{aligned} \tag{2.18}$$

The canonical transformations (2.18) are illustrated in Fig. 2.4-a. After the transformations, the two new but equivalent canonical sets $\{\bar\sigma_1(l_1) = \mathcal{G}(B); \bar\sigma_3(l_1)\}$, $\{\sigma_1(l_{12}); \sigma_3(l_{12})\}$ are attached to the links l_1 and $l_{12} \equiv l_1 l_2$ respectively. They satisfy the same commutation relations as the original operators (2.8):

$$\begin{aligned} \bar\sigma_1(l_1)\bar\sigma_3(l_1) + \bar\sigma_3(l_1)\bar\sigma_1(l_1) = 0, \\ \sigma_1(l_{12})\sigma_3(l_{12}) + \sigma_3(l_{12})\sigma_1(l_{12}) = 0. \end{aligned} \tag{2.19}$$

One can easily check: $(\bar{\sigma}_1(l_1))^2 = 1$, $(\bar{\sigma}_3^2(l_1))^2 = 1, (\sigma_1(l_{12}))^2 = 1, (\sigma_3(l_{12}))^2 = 1$. Further, note that the two conjugate pairs $\{\bar{\sigma}_1(l_1); \bar{\sigma}_3(l_1)\}$ and $\{\sigma_1(l_1l_2); \sigma_3(l_1l_2)\}$ are also mutually independent as they commute with each other. As an example, $[\bar{\sigma}_1(l_1), \sigma_3(l_1l_2)] \equiv [\sigma_1(l_1)\sigma_1(l_2), \sigma_3(l_1)\sigma_3(l_2)] = 0$. The new conjugate pair $\{\bar{\sigma}_1(l_1); \bar{\sigma}_3(l_1)\}$ is frozen due to the Gauss law at B: $\bar{\sigma}_1(l_1) = \mathcal{G}(B) \approx 1$ in $\mathcal{H}^{\text{phys}}$. We now repeat (2.18) with l_1, l_2 replaced by l_{12}, l_3 respectively to define new conjugate operators $\{\sigma_1(l_{123}); \sigma_3(l_{123})\}$ and $\{\bar{\sigma}_1(l_3); \bar{\sigma}_3(l_3)\}$ attached to the links $l_{123}(\equiv l_1l_2l_3)$ and l_3 respectively:

$$\begin{aligned} \sigma_3(l_{123}) &\equiv \sigma_3(l_{12})\sigma_3(l_3), & \bar{\sigma}_3(l_3) &\equiv \sigma_3(l_3) \\ \sigma_1(l_{123}) &= \sigma_1(l_{12}) = \sigma_1(l_2), & \bar{\sigma}_1(l_3) &= \sigma_1(l_{12})\sigma_1(l_3) = \mathcal{G}(C) \end{aligned} \tag{2.20}$$

As before, the new conjugate pair $\{\bar{\sigma}_1(l_3);\ \bar{\sigma}_3(l_3)\}$ becomes unphysical as $\bar{\sigma}_1(l_3) = \mathcal{G}(C) \approx 1$ in $\mathcal{H}^{\text{phys}}$. The last canonical transformations involve gluing the conjugate pairs $\{\sigma_1(l_{123});\ \sigma_3(l_{123})\}$ with $\{\sigma_1(l_4);\ \sigma_3(l_4)\}$ to define the dual and gauge invariant plaquette variables $\{\mu_1(p); \mu_3(p)\}$, with $p \equiv l_1l_2l_3l_4$:

$$\mu_1(p) \equiv \sigma_3(l_{123})\sigma_3(l_4) \equiv \sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4), \qquad \mu_3(p) \equiv \sigma_1(l_{123}) = \sigma_1(l_2). \tag{2.21a}$$

$$\bar{\sigma}_3(l_4) \equiv \sigma_3(l_4), \qquad \bar{\sigma}_1(l_4) = \sigma_1(l_{123})\sigma_1(l_4) = \sigma_1(l_2)\sigma_1(l_4) \equiv \mathcal{G}(C)\mathcal{G}(D). \tag{2.21b}$$

To summarize, the three canonical transformations (2.18), (2.20), (2.21a) and (2.21b) transform the initial four conjugate sets $\{\sigma_1(l_1); \sigma_3(l_1)\}$, $\{\sigma_1(l_2); \sigma_3(l_2)\}$, $\{\sigma_1(l_3); \sigma_3(l_3)\}$, $\{\sigma_1(l_4); \sigma_3(l_4)\}$ attached to the links l_1, l_2, l_3, l_4 to four new and equivalent canonical sets $\{\bar{\sigma}_1(l_1); \bar{\sigma}_3(l_1)\}$, $\{\bar{\sigma}_1(l_3); \bar{\sigma}_3(l_3)\}$, $\{\bar{\sigma}_1(l_4); \bar{\sigma}_3(l_4)\}$ and $\{\mu_1(p); \mu_3(p)\}$ attached to the links l_1, l_3, l_4 and the plaquette p respectively. The advantage of the new sets is that all the three independent Gauss law constraints at B, C and D are automatically solved. They freeze the three strings leaving us only with the physical spin or plaquette loop conjugate operators $\{\mu_1(p); \mu_3(p)\}$. The defining canonical relations (2.18), (2.20), (2.21a) and (2.21b) can also be inverted. The inverse transformations from the new spin flux operators to Z_2 link flux operators are

$$\begin{aligned} \sigma_3(l_1) &= \bar{\sigma}_3(l_1), & \sigma_3(l_3) &= \bar{\sigma}_3(l_3), \\ \sigma_3(l_3) &= \bar{\sigma}_3(l_1)\mu_1(p)\bar{\sigma}_3(l_3)\bar{\sigma}_3(l_4), & \sigma_3(l_4) &= \bar{\sigma}_3(l_4). \end{aligned} \tag{2.22}$$

Similarly, the initial conjugate Z_2 electric field operators on the links are

$$\begin{aligned} \sigma_1(l_1) &= \mu_3(p)\ \bar{\sigma}_1(l_1) = \mu_3(p)\ \mathcal{G}(B) \approx \mu_3(p), \\ \sigma_1(l_2) &= \mu_3(p) \\ \sigma_1(l_3) &= \mu_3(p)\ \bar{\sigma}_1(l_3) = \mu_3(p)\ \mathcal{G}(C) \approx \mu_3(p) \\ \sigma_1(l_4) &= \mu_3(p)\ \bar{\sigma}_1(l_4) = \mu_3(p)\ \mathcal{G}(C)\mathcal{G}(D) \approx \mu_3(p). \end{aligned} \tag{2.23}$$

Thus the complete set of gauge-spin duality relations over a plaquette and their inverses are given in (2.18), (2.20), (2.21a), (2.21b) and (2.22), (2.23) respectively. Note that the Gauss law constraint at site A does not play any role as $\mathcal{G}(A) \approx \mathcal{G}(B)\ \mathcal{G}(C)\ \mathcal{G}(D)$. The total number of degrees of freedom also match. The initial Z_2 gauge theory had 4 spins with 3 Gauss law constraints. In the final dual spin model the 3 gauge non-invariant strings take care of the 3 Gauss law constraints leaving us with the single gauge invariant spin described by $\{\mu_1(p); \mu_3(p)\}$ on the plaquette p. The single plaquette Z_2 lattice gauge theory Hamiltonian (2.11) can now be rewritten in terms of the new gauge invariant spins as:

$$H \approx -4\ \mu_3(p) - \lambda\ \mu_1(p) = -\begin{pmatrix} \lambda & 4 \\ 4 & -\lambda \end{pmatrix}. \tag{2.24}$$

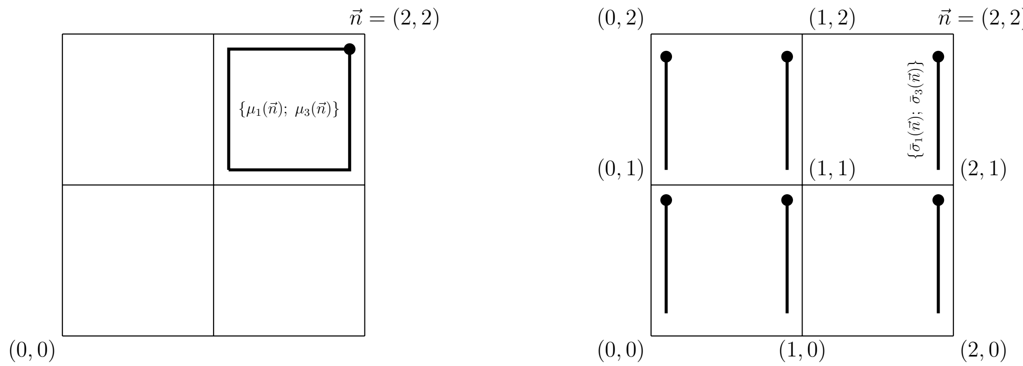
Note that the equivalence of the gauge and spin Hamiltonians (2.11) and (2.24) respectively is valid only within the physical Hilbert space $\mathcal{H}^{\text{phys}}$. The two energy eigenvalues of H are $\epsilon_{\pm} = \pm 4\sqrt{\left(1 + \left(\frac{\lambda}{4}\right)^2\right)}$.

Having discussed the essential ideas, we now directly write down the general Z_2 gauge-spin duality or canonical relations by iterating the canonical transformations (2.18), (2.20), (2.21a) and (2.21b) over the entire lattice. Note that there are $\mathcal{L}$ initial spins (one on every link) with $\mathcal{N}$ Gauss law constraints (one at every site) satisfying the identity:

$$\prod_{(m,n)\in\Lambda} \mathcal{G}(m,n)\ \equiv\ 1. \tag{2.25}$$

The above identity again states that simultaneous flipping of all spins around every lattice site is an identity operator because each spin is flipped twice. As mentioned earlier, it is a property of all Abelian gauge theories which reduces the number of Gauss law constraints from $\mathcal{N}$ to $(\mathcal{N}-1)$. After canonical transformations in Z_2 lattice gauge theory, there are (a) $\mathcal{P}$ physical plaquette spins (analogous to $\{\mu_1(p); \mu_3(p)\}$ in the single plaquette case) shown in Figure 2.5-a and (b) $(\mathcal{N}-1)$ string spins (analogous to $\{\bar{\sigma}_1(l_1); \bar{\sigma}_3(l_1)\}$; $\{\bar{\sigma}_1(l_3); \bar{\sigma}_3(l_3)\}$ and $\{\bar{\sigma}_1(l_4); \bar{\sigma}_3(l_4)\}$ in the single plaquette case) as every lattice site away from the origin can be attached to a unique string. This is shown in Figure 2.5-b. The degrees of freedom before and after the canonical transformations match as $\mathcal{L} = \mathcal{P} + (\mathcal{N}-1)$. All $(\mathcal{N}-1)$ strings decouple because of the $(\mathcal{N}-1)$ Gauss law constraints.

From now onward the $\mathcal{P}$ physical plaquette spin/loop operators are labelled by the top right corners of the corresponding plaquettes as shown in Figure 2.5-a). The vertical (horizontal) stringy spin operators are labelled by the top (right) end points of the corresponding links as shown in Figure 2.5-b.



(a) Z_2 spins/loops (physical) (b) Z_2 string (unphysical)

Figure 2.5: The physical Z_2 spin conjugate pairs $\{\mu_1(\vec{n}); \mu_3(\vec{n})\}$ and the unphysical string conjugate pairs $\{\bar{\sigma}_1(\vec{n}); \bar{\sigma}_3(\vec{n})\}$ dual to Z_2 lattice gauge theory are shown in (a) and (b) respectively. The co-ordinates of spin or loop operators are the co-ordinates of their top right corners. The co-ordinates of the horizontal (vertical) strings are the co-ordinates of their right (top) end points. These are shown by • in (a) and (b). The strings decouple from the physical Hilbert space as $\bar{\sigma}_1(\vec{n}) = \mathcal{G}(\vec{n}) \approx 1$ by Gauss law constraint at $\vec{n}$.

Physical sector and Z_2 dual potentials

The final duality relations between the initial conjugate sets $\{\sigma_1(m,n;\hat{i}); \sigma_3(m,n;\hat{i})\}$ on every lattice link $(m,n;\hat{i})$ and the final physical conjugate loop operators $\{\mu_1(m,n);\ \mu_3(m,n)\}$ are

$$\mu_1(m,n) = \sigma_3(m-1,n-1;\hat{1})\ \sigma_3(m-1,n-1;\hat{2})\sigma_3(m,n;-\hat{2})\sigma_3(m,n;-\hat{1}), \tag{2.26a}$$

$$\mu_3(m,n) = \prod_{n'=n}^{N} \sigma_1(m-1,n';\hat{1}). \tag{2.26b}$$

In (2.26a) we have defined $\sigma_1(m,n;-\hat{1}) \equiv \sigma_1(m-1,n;\hat{1})$ and $\sigma_1(m,n;-\hat{2}) \equiv \sigma_1(m,n-1;\hat{2})$. The relations (2.26a) and (2.26b) are the extension of the single plaquette relations (2.21a) to the entire lattice. They are illustrated in Figure 2.6-a. The canonical commutation relations are

$$\mu_1(m,n)\mu_3(m,n) + \mu_3(m,n)\mu_1(m,n) = 0. \tag{2.27}$$

Further, $(\mu_3(m,n))^2 = 1, (\mu_1(m,n))^2 = 1$. *The canonical transformations (2.26a) and (2.26b) are important as they define the magnetic field operators $\mu_1(m,n)$ and its conjugate $\mu_3(m,n)$ as new dual fundamental operators. The electric field is derived from the electric scalar potentials.* This should be contrasted with the original description where electric fields $\sigma_1(m,n)$ were fundamental and the magnetic fields were derived from the magnetic magnetic vector potentials as $\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4)$.

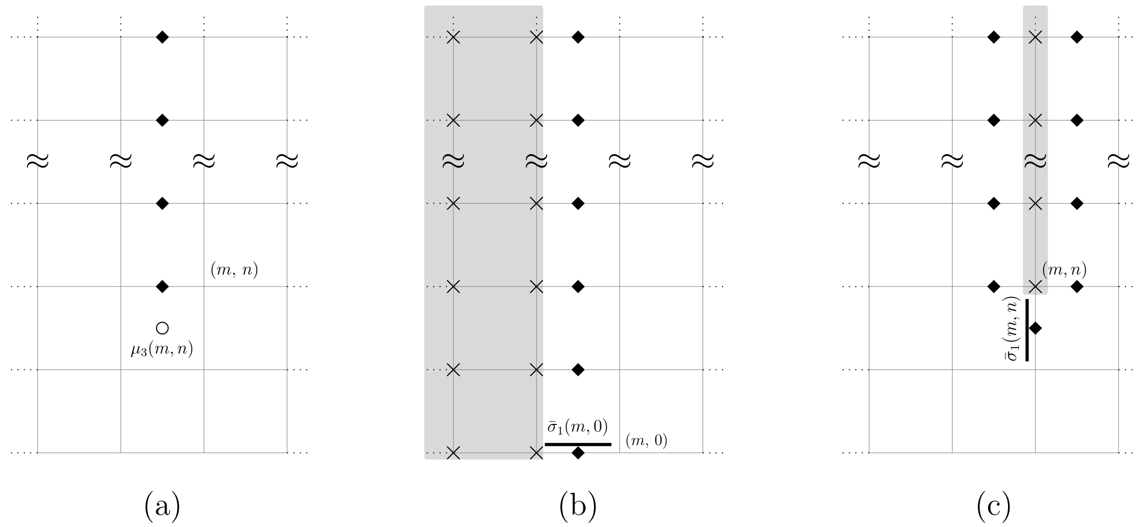


Figure 2.6: The non-local relations in the Z_2 gauge-spin duality transformations: (a) shows the relations (2.26b) expressing $\mu_3(m,n)$ as the product of σ_1 operators denoted by ♦. In (b) and (c), we show the relations (2.28b) expressing $\bar{\sigma}_1(m,0)$ and $\bar{\sigma}_1(m,n)$; $n \neq 0$ respectively as the product of σ_1 operators denoted by ♦. As $\sigma_1^2 = 1$, the string operators $\bar{\sigma}_1(m,0)$ and $\bar{\sigma}_1(m,n)$ are also a product of Gauss law operators at sites marked by $\times$ in the shaded regions. For similar SU(N) relations, see Figures 4.3.

Unphysical sector and Z_2 string operators

The unphysical string conjugate pair operators are

$$\begin{aligned} \bar{\sigma}_3(m,0) &= \sigma_3(m-1,0;\hat{1}), \\ \bar{\sigma}_3(m,n) &= \sigma_3(m,n-1;\hat{2}); \qquad n \neq 0 \end{aligned} \tag{2.28a}$$

$$\begin{aligned} \bar{\sigma}_1(m,0) &= \prod_{m'=0}^{m-1}\prod_{n'=0}^{N} \mathcal{G}(m',n') \approx 1, \\ \bar{\sigma}_1(m,n) &= \prod_{n'=n}^{N} \mathcal{G}(m,n') \approx 1; \qquad n \neq 0. \end{aligned} \tag{2.28b}$$

The relations (2.28a) and (2.28b) are illustrated in Figure 2.6-b and Figure 2.6-c respectively. It is easy to see that in the full gauge theory Hilbert space $\bar{\sigma}_1(m,n)\bar{\sigma}_3(m,n)+\bar{\sigma}_3(m,n)\bar{\sigma}_1(m,n) = 0$ and different string operators located at different lattice sites commute with each others. Further, one can check that all strings and plaquette operators are mutually independent and commute with each other:

$$[\mu_3(m,n),\ \bar{\sigma}_1(m',n')] = 0, \qquad [\mu_3(m,n),\ \bar{\sigma}_3(m',n')] = 0, \tag{2.29}$$

$$[\mu_1(m,n), \bar{\sigma}_1(m',n')] = 0, \qquad [\mu_1(m,n), \bar{\sigma}_3(m',n')] = 0.$$

Inverse relations

The inverse relations for the flux operators over the entire lattice are

$$\begin{aligned}
\sigma_3(m,0;\hat{1}) &= \bar{\sigma}_3(m+1,0), \\
\sigma_3(m,n;\hat{2}) &= \bar{\sigma}_3(m,n+1) \\
\sigma_3(m,n;\hat{1}) &= \left(\prod_{l=1}^{n} \bar{\sigma}_3(m,l)\right)\left(\prod_{q=1}^{n} \bar{\sigma}_3(m+1,q)\right)\left(\prod_{p=1}^{n} \mu_1(m+1,p)\right); \quad n \neq 0
\end{aligned} \tag{2.30}$$

On the other hand, the conjugate electric field operators are

$$\begin{aligned}
\sigma_1(m,n;\hat{1}) &= \mu_3(m,n)\mu_3(m,n+1), \\
\sigma_1(m,n;\hat{2}) &= \mu_3(m,n+1)\mu_3(m+1,n+1).
\end{aligned} \tag{2.31}$$

In the second relation in (2.31), we have used Gauss laws at (m,l) ; $l = n+1, n+2, \cdots$. The above relations are analogous to the inverse relations (2.22) and (2.23) in the single plaquette case.

Z_2 Gauss laws & solutions

It is easy to see that the Gauss law constraints are automatically satisfied by the dual spin operators as shown in Figure 2.7-a,b. We write the Z_2 electric fields around a site (m,n) in terms of the electric scalar potentials:

$$\begin{aligned}
\sigma_1(l_1) &\equiv \sigma_1(m,n;\hat{1}) = \mu_3(p_1)\mu_3(p_2), &\quad \sigma_1(l_2) &\equiv \sigma_1(m,n;\hat{2}) = \mu_3(p_2)\mu_3(p_3), \\
\sigma_1(l_3) &\equiv \sigma_1(m-1,n;\hat{1}) = \mu_3(p_3)\mu_3(p_4), &\quad \sigma_1(l_4) &\equiv \sigma_1(m,n-1;\hat{2}) = \mu_3(p_4)\mu_3(p_1).
\end{aligned} \tag{2.32}$$

In (2.32) we have used link and plaquette labels from Figure 2.7. As $(\mu_3(p))^2 = 1$, we get

$$\mathcal{G}(m,n) = \sigma_1(l_1)\sigma_1(l_2)\sigma_1(l_3)\sigma_1(l_4) \equiv 1. \tag{2.33}$$

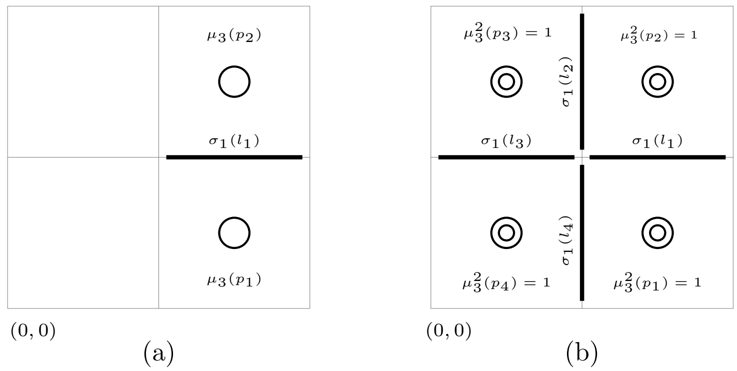


Figure 2.7: (a) shows the Z_2 link electric field operator $\sigma_1(l_1) \equiv \sigma_1(m,n;\hat{1})$ as the product of nearest neighbour loop operators $\mu_3(p_1)$ and $\mu_3(p_2)$, (b) graphically illustrates how the spin or electric potential operators $\{\mu_3(p_1), \mu_3(p_2), \mu_3(p_3), \mu_3(p_4)\}$ solve the Z_2 Gauss law (2.33) at site (m,n).

Z_2 dual dynamics

The Z_2 lattice gauge theory Hamiltonian (2.11) in terms of the physical spin operators takes the simple nearest neighbour interaction form:

$$\begin{aligned} H &= -\sum_{<p,p'>} \mu_3(p)\mu_3(p') - \lambda \sum_p \mu_1(p) \equiv H_E + \lambda H_B, \\ &= \lambda \Big[-\sum_p \mu_1(p) - \frac{1}{\lambda} \sum_{<p,p'>} \mu_3(p)\mu_3(p') \Big] \end{aligned} \tag{2.34}$$

In (2.34) $\sum_{<p,p'>}$ denotes the sum over the nearest neighbor plaquettes. Note that the original fundamental non-interacting electric field terms are now described by nearest neighbour interacting electric scalar potentials. The non-interacting magnetic fields, on the other hand, have now acquired fundamental status. Thus the two gauge-spin descriptions:

$$\{\sigma_1(l); \sigma_3(l)\} \longleftrightarrow \{\mu_1(p); \mu_3(p)\}$$

are related by duality. Further, Z_2 lattice gauge theory at coupling λ is mapped into Z_2 spin model at coupling $(1/\lambda)$, i.e,

$$H^{Z_2}_{gauge}(\lambda) \simeq \lambda \; H^{Z_2}_{spin}(1/\lambda).$$

We have used $\simeq$ above to emphasizes that this equivalence is only within the physical Hilbert space $\mathcal{H}^{\rm phys}$.

2.2.1 Z_2 Magnetic disorder operator

The dual spin model (2.34) on an infinite lattice has global Z_2 invariance:

$$\mu_1(p) \to \mu_1(p), \quad \mu_3(p) \to -\mu_3(p), \quad \forall p \in \Lambda. \tag{2.35}$$

Its generator $G_\Lambda \equiv \prod_{p\in\Lambda} \mu_1(p)$ leaves the Hamiltonian (2.34) invariant: $G_\Lambda H G_\Lambda^{-1} = H$. Unlike the initial Z_2 gauge symmetry of Z_2 gauge theory, the global Z_2 symmetry of the dual spin model (2.34) is the symmetry of the spectrum. Being independent of gauge invariance, it allows the Ising spin model (2.34) to be magnetized through spontaneous symmetry breaking for $\lambda << 1$. As a consequence of duality:

$$\begin{aligned}
\Big\langle \mu_1(p) \Big\rangle_{H^{z2}_{spin}(1/\lambda)} &= \Big\langle \sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4) \Big\rangle_{H^{z2}_{gauge}(\lambda)} \\
\Big\langle \mu_3(m,n) \Big\rangle_{H^{z2}_{spin}(1/\lambda)} &= \Big\langle \prod_{n'=n}^{N} \sigma_1(m,n') \Big\rangle_{H^{z2}_{gauge}(\lambda)}
\end{aligned} \tag{2.36}$$

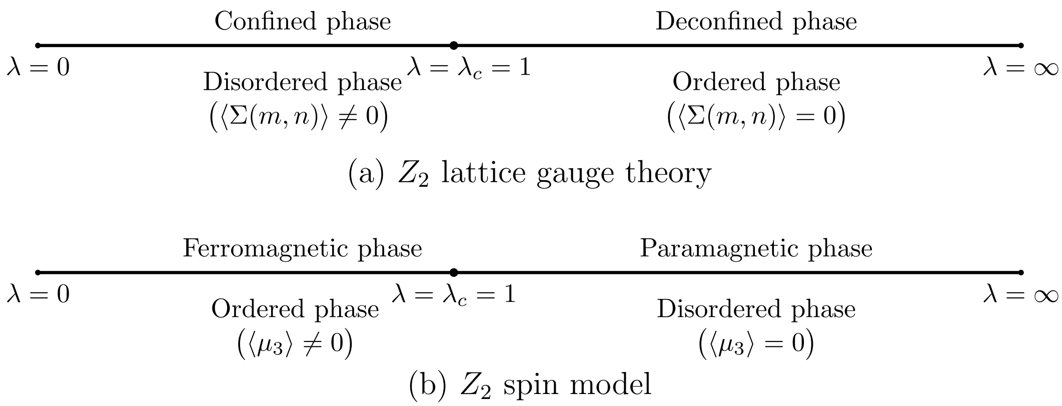
The above two equations describe the relationship between order and disorder in the gauge and the dual spin system. Note that we always measure order or disorder with respect to the potentials. The first relation above states that at low temperature or large coupling $\lambda >> 1$, the gauge system is in ordered phase. This is because all magnetic vector potentials ($\sigma_3(l)$) are aligned (close to unity) leading to $\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4) \approx 1$. This is the free phase of Z_2 gauge theory mentioned in the introduction with Wilson loop following perimeter law:

$$\langle W_{[\mathcal{C}]} \rangle \equiv \langle \prod_{l\in\mathcal{C}} \sigma_3(l) \rangle = \exp\Big(- \lambda^{-2} \ \text{Perimeter}(\mathcal{C})\Big), \qquad \lambda >> 1. \tag{2.37}$$

However, the dual spin system is now at high temperature. It is in the disordered phase as the dual electric scalar potential or the spin values $\mu_3(p) = \pm 1$ are equally probable. On the other hand, at small coupling ($\lambda << 1$), the spin system is ordered with all electric scalar potentials aligned to the value $\mu_3(p) = +1$ or -1. The gauge system is now disordered as the two values of the magnetic vector potentials $\sigma_3(l) = \pm 1$ are equally probable. This is the confining phase with the Z_2 Wilson loop $W_{[\mathcal{C}]}$ around a closed curve $\mathcal{C}$ following the area law:

$$\langle W_{[\mathcal{C}]} \rangle \equiv \langle \prod_{l\in\mathcal{C}} \sigma_3(l) \rangle \sim (\lambda)^{\text{Area}(\mathcal{C})} = \exp\Big(- |\ln \lambda| \ \text{Area}(\mathcal{C})\Big), \qquad \lambda << 1. \tag{2.38}$$

The disorder in the gauge system is the order in the dual spin system which is measured by the expectation value of electric scalar potential $\mu_3(p)$. It is a (non-local) product of the original link electric fields which flip the magnetic vector potentials $\sigma_3(l)$ along an infinite path.



(a) Z_2 lattice gauge theory

(b) Z_2 spin model

Figure 2.8: Duality and order, disorder in (a) $(2+1)$ dimensional Z_2 lattice gauge theory, (b) $(2+1)$ dimensional Ising model. The confining ($\lambda << 1$) and deconfining ($\lambda >> 1$) phases of Z_2 lattice gauge theory correspond to the ferromagnetic and paramagnetic phases of the Ising spin model [3, 29, 37, 40, 94, 98, 111–113].

This is shown in Figure 2.6-a. For latter convenience and comparisons with SU(N) results, the Z_2 disorder operator is relabeled as:

$$\Sigma(m,n) \equiv \mu_3(m,n) = \prod_{n'=n}^{N} \sigma_1(m,n'). \tag{2.39}$$

Just like in the case of Kramers-Wannier duality [3, 36, 37, 40, 94, 98, 111–113, 127], the disorder operator $\Sigma(m,n)$ in Z_2 gauge theory acting on an ordered state creates a kink state [3,37] which is orthogonal to the original ordered state. Note that a kink at plaquette p is a magnetic vortex at p in the original gauge language. Therefore the expectation value of the disorder operator in an ordered state (no kinks or vortices) is 0. Below the critical point λ_c, its expectation value is non-zero. This is the disordered phase and can be understood in terms of kink or magnetic vortex condensation [3]. We therefore obtain:

$$\langle \Sigma(m,n) \rangle \neq 0 \qquad \lambda << 1, \qquad \langle \Sigma(m,n) \rangle \; = 0 \qquad \lambda >> 1. \tag{2.40}$$

The Z_2 gauge-spin duality and the phase diagrams are shown in Figure 2.8. Note that the disorder operator $\Sigma(m,n)$ is gauge invariant as it commutes with the local Gauss law operators $\mathcal{G}(n)$. We further define $\mu_3(m,n) \equiv e^{i\pi\mathcal{E}(m,n)}$. Using (2.9), we get:

$$\Sigma(m,n) = \exp\; i\Big(\pi\; \mathcal{E}(m,n)\Big) = \exp\; i\Big(\pi \sum_{n'=n}^{N} E(m,n';\hat{1})\Big) \equiv \Sigma_\pi(m,n). \tag{2.41}$$

The order-disorder algebra is obtained by using the anti-commutation relation between $\sigma_1(l)$

and $\sigma_3(l)$:

$$W_{[\mathcal{C}]}\ \Sigma_\pi(m,n) = (-1)^q\ \Sigma_\pi(m,n)\ W_{[\mathcal{C}]}. \tag{2.42}$$

As $\mathcal{C}$ is a closed loop: $q = 1$ if the point (m,n) is inside $\mathcal{C}$ and $q = 0$ if (m,n) is outside $\mathcal{C}$. This can be generalized to more complicated curves where q equals the winding number which is the number of times the curve $\mathcal{C}$ winds around the plaquette at (m,n). The algebra (2.42) is the standard Wilson-'t Hooft loop algebra for the simplest Z_2 lattice gauge theory in $(2+1)$ dimensions.

2.3 Electromagnetic duality in U(1) lattice gauge theory

The compact U(1) lattice gauge theory is defined as

$$H = g^2 \sum_l E^2(l) + \frac{K}{g^2}\sum_p (2 - U_p - U_p^\dagger) \tag{2.43}$$

In (2.43), $U(l)$ is the Abelian (phase factor) holonomy satisfying $U^\dagger(l)U(l) = U(l)U^\dagger(l) = 1$ on link l, g is lattice coupling and K is a constant. The plaquette operators $U_p = \prod_{l_p \in p} U(l_p)$ is a product of four-link operators around a plaquette p. This is an electric field $E(l)$ & magnetic vector potential $U(l)$ description. The link conjugate pair $(E(l), U(l))$ satisfies canonical commutations relations:

$$[E(l), U(l)] = U(l), \qquad [E(l), U^\dagger(l)] = -U^\dagger(l) \tag{2.44}$$

The U(1) gauge transformations at site $\vec{n}$ these link operators and electric fields transform as:

$$U(\vec{n};\hat{i}) \rightarrow \Lambda(\vec{n})\ U(\vec{n};\hat{i})\ \Lambda^\dagger(\vec{n}+\hat{i}) \tag{2.45a}$$

$$E(\vec{n};\hat{i}) \rightarrow \quad E(\vec{n};\hat{i}) \tag{2.45b}$$

The canonical commutation relations (2.44) imply that the generator of the above gauge transformations are

$$\mathcal{G}(\vec{n}) = \sum_{i=1}^{2}\Big(E(\vec{n};\hat{i}\ ) - E(\vec{n}-\hat{i};\hat{i}\ )\Big) \tag{2.46}$$

The physical Hilbert space of the theory can be span by the state $|\text{phys}\rangle$ which satisfies Gauss law constraints (2.46).

$$\mathcal{G}(\vec{n})\ |\text{phys}\rangle = 0, \qquad \forall\ \vec{n}. \tag{2.47}$$

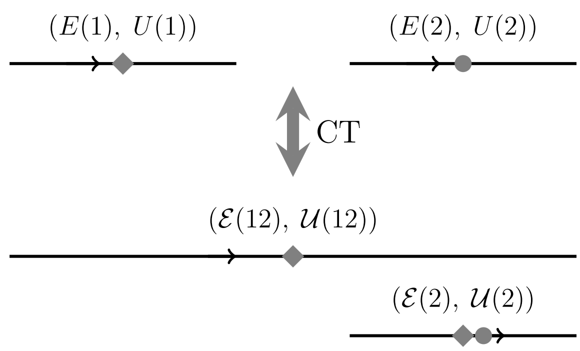


Figure 2.9: Fundamental canonical transformation for gluing two link conjugate pairs $(E(1),\ U(1))$ and $(E(2),\ U(2))$. Two new mutually independent holonomies are $(\mathcal{E}(12),\ \mathcal{U}(12))$ and $(\mathcal{E}(2),\ \mathcal{U}(2))$. The electric field of U(1), denoted by ♦ which has been traded off for a bigger holonomy $\mathcal{U}(12)$ appear in both the new holonomies.

2.3.1 Canonical transformations: U(1) duality

In this section, we will obtain the electromagnetic duality relations in compact U(1) gauge theory by means of canonical transformations. For simplicity we first consider a single plaquette case shown in Figure 2.10. The fundamental canonical transformations for gluing two link conjugate pairs are explained in the Figure 2.9. Two link pairs $(E(1),\ U(1))$ and $(E(2),\ U(2))$ can be canonically glued to produce two new canonical pairs $(\mathcal{E}(12),\ \mathcal{U}(12))$ and $(\mathcal{E}(2),\ \mathcal{U}(2))$:

$$
\begin{aligned}
\mathcal{U}(12) &\equiv U(1)U(2), & \mathcal{E}(12) &= \quad E(1) \\
\mathcal{U}(2) &\equiv \quad U(2), & \mathcal{E}(2) &= -E(1) + E(2)
\end{aligned}
\tag{2.48}
$$

These two canonical pairs satisfy the same commutation relation

$$
[\mathcal{E}(12), \mathcal{U}(12)] = \mathcal{U}(12), \qquad [\mathcal{E}(2), \mathcal{U}(2)] = \mathcal{U}(2) \tag{2.49}
$$

and they are mutually independent

$$
[\mathcal{E}(12), \mathcal{U}(2)] = 0, \qquad [\mathcal{E}(2), \mathcal{U}(12)] = 0 \tag{2.50}
$$

Three such canonical transformations of four-link conjugate pairs around a plaquette produce one plaquette and its conjugate scalar electric field along with other three string holonomies, see Figure 2.10. Using fundamental canonical transformation (2.48), we can construct canonical transformation for four link pairs $(E(1), U(1))$, $(E(2), U(2))$, $(E(3), U(3))$ and $(E(4), U(4))$ producing a plaquette loop $(\mathcal{E}(p), \mathcal{W}(p))$ and three string pairs $(\mathcal{E}(1), \mathcal{U}(1))$, $(\mathcal{E}(3), \mathcal{U}(3))$ and $(\mathcal{E}(4), \mathcal{U}(4))$;

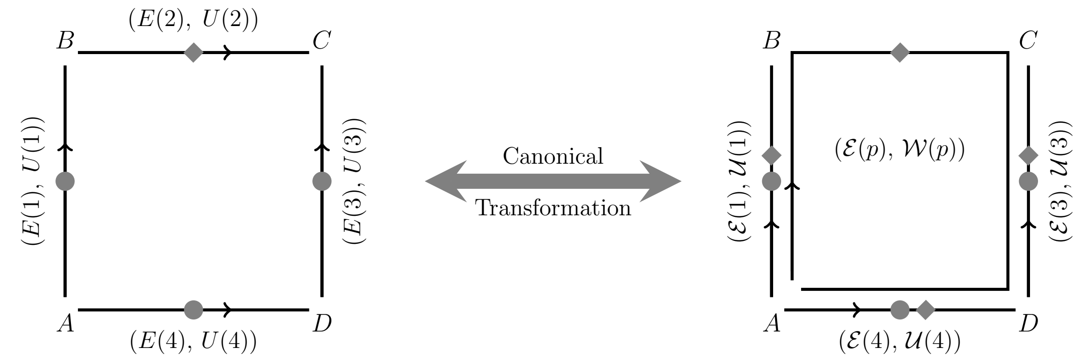


Figure 2.10: The U(1) canonical transformations over single plaquette. Four link conjugate pairs $(E(l),\, U(l)), l = 1,2,3,4$ are converted to four new mutually independent degrees of freedom.

$$\begin{aligned} \mathcal{U}(1) &\equiv U(1), & \mathcal{E}(1) &= E(1) - E(2) \\ \mathcal{W}(p) &\equiv U(1)U(2)U^\dagger(3)U^\dagger(4), & \mathcal{E}(p) &= E(2) \\ \mathcal{U}(3) &\equiv U(3), & \mathcal{E}(3) &= E(3) + E(2) \\ \mathcal{U}(4) &\equiv U(4), & \mathcal{E}(4) &= E(4) + E(2) \end{aligned} \tag{2.51}$$

Note that the magnetic fields described by $\mathcal{W}(p)$ and their conjugate dual electric scalar potentials $\mathcal{E}(p)$. The remaining conjugate pairs $(\mathcal{E}(1),\, \mathcal{U}(1))$, $(\mathcal{E}(3),\, \mathcal{U}(3))$ and $(\mathcal{E}(4),\, \mathcal{U}(4))$ are the unphysical gauge degrees of freedom.

Inverse Canonical Transformations & U(1) Gauss Laws

The canonical transformations (2.51) can be easily inverted to get link operators and their electric fields

$$\begin{aligned} U(1) &= \mathcal{U}(1), & E(1) &= \mathcal{E}(1) + \mathcal{E}(p) \\ U(2) &= \mathcal{U}^\dagger(1)\mathcal{W}(p)\mathcal{U}(3)\mathcal{U}(4), & E(2) &= \mathcal{E}(p) \\ U(3) &= \mathcal{U}(3), & E(3) &= \mathcal{E}(3) - \mathcal{E}(p) \\ U(4) &= \mathcal{U}(4), & E(4) &= \mathcal{E}(4) - \mathcal{E}(p) \end{aligned} \tag{2.52}$$

Note that the 4 original Gauss law constraints at the 4 lattice sites A,B,C & D turn into the 4 dual Gauss laws satisfied by the dual electric scalar potentials:

1. Gauss law at A: $\mathcal{G}(A) = E(1) + E(4) = \mathcal{E}(1) + \mathcal{E}(4)$

2. Gauss law at B: $\mathcal{G}(B) = E(2) - E(1) = -\mathcal{E}(1)$

3. Gauss law at C: $\mathcal{G}(C) = -(E(2) + E(3)) = -\mathcal{E}(3)$
4. Gauss law at D: $\mathcal{G}(D) = E(3) - E(4) = \mathcal{E}(3) - \mathcal{E}(4)$

We also conclude that the three strings are auxiliary as their conjugate electric fields vanish in $\mathcal{H}^{\text{phys}}$:

$$\begin{aligned}\mathcal{E}(1) &= -\,\mathcal{G}(A) \sim 0\\ \mathcal{E}(3) &= -\,\mathcal{G}(C) \sim 0\\ \mathcal{E}(4) &= -\,\mathcal{G}(C) - \mathcal{G}(D) \sim 0\end{aligned} \tag{2.53}$$

2.3.2 $N \times N$ Lattice

Now we consider a finite $N \times N$ lattice with free boundary conditions. We start making canonical transformations at the top left most plaquette and proceed to the plaquette below, continuing until we reach the bottom of the column. We then repeat the same process in the right adjacent column and continue this process until we reach the right most column. Note that initially there are $2N(N+1)$ horizontal and vertical holonomies. Using canonical transformations we convert them into N^2 physical plaquette holonomies along with $N(N+1)$ vertical and N horizontal holonomies. Thus the total number of degrees of freedom is preserved. As at each plaquette, there are 3 canonical transformations required, at the end of $3N^2$ canonical transformations, we get the following dual degrees of freedom:

[A] N^2 **plaquette fluxes**

$$\mathcal{W}(\vec{n}) = U(\vec{n};\hat{2})U(\vec{n}+\hat{2};\hat{1})U^{\dagger}(\vec{n}+\hat{1};\hat{2})U^{\dagger}(\vec{n};\hat{1}) \tag{2.54}$$

and their conjugate electric fields are

$$\mathcal{E}(m,n) = \sum_{j=n+1}^{N} E(m,j;\hat{1}) \tag{2.55}$$

The above canonical relation is pictorially shown in Figure 2.11. The plaquette fluxes and their conjugate scalar electric potentials obey standard canonical commutation relations (2.44)

$$[\mathcal{E}(\vec{n}),\ \mathcal{W}(\vec{n})] = \mathcal{W}(\vec{n}) \tag{2.56}$$

The U(1) gauge transformations (2.45a), these new plaquette operators remain invariant and therefore act like spin degrees of freedom.

$$\mathcal{W}(\vec{n}) \rightarrow \mathcal{W}(\vec{n}), \qquad\qquad \mathcal{E}(\vec{n}) \rightarrow \mathcal{E}(\vec{n}) \tag{2.57}$$

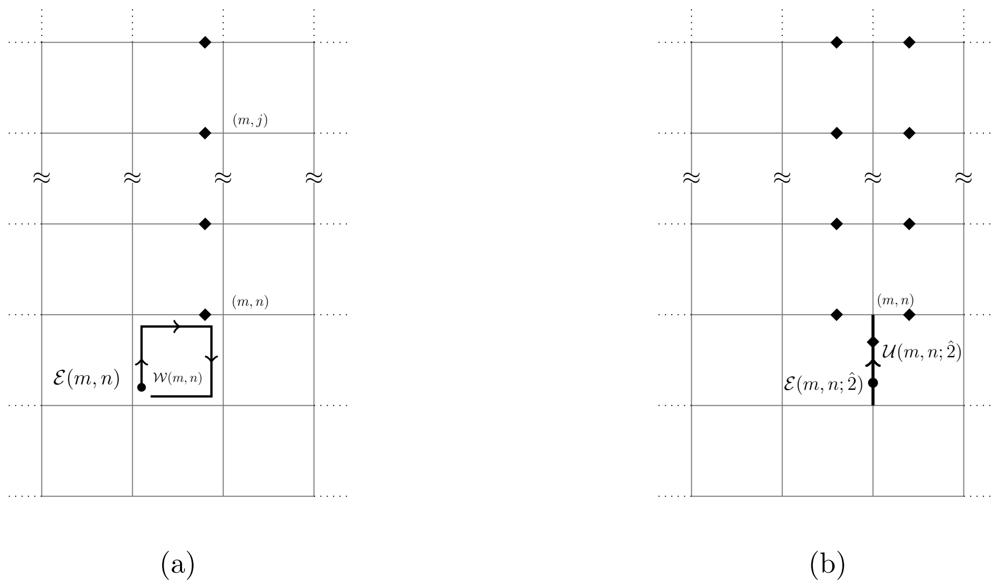


Figure 2.11: Non-local duality relations between Kogut-Susskind electric fields and dual scalar electric potentials: (a) shows an expression of plaquette electric potential $\mathcal{E}(m,n)$ (denoted by $\bullet$) as a sum of Kogut-Susskind electric fields $E(m, j > n; \hat{1})$ as in equation (2.55) (b) shows the expression of vertical string's electric field as in equation (2.59). Kogut-Susskind electric fields are denoted by ⧫ on corresponding links.

[B] $N(N+1)$ **vertical string fluxes**

$$\mathcal{U}(\vec{n};\hat{2}) = U(\vec{n};\hat{2}) \tag{2.58}$$

and their conjugate electric fields are

$$\mathcal{E}(m,n;\hat{2}) = E(m,n;\hat{2}) - \sum_{j=n+1}^{N} \big(E(m,j;\hat{1}) - E(m-1,j;\hat{1})\big) \tag{2.59}$$

These vertical string fluxes are unphysical or cyclic as they can not be part of any gauge invariant operator and therefore do not appear in the dynamics. Infact, their electric fields vanish in physical Hilbert space. We can see this by noticing that in (2.59)

$$E(m,n;\hat{2}) - \sum_{j=n+1}^{N} \big(E(m,j;\hat{1}) - E(m-1,j;\hat{1})\big) = -\sum_{j=n+1}^{N} \mathcal{G}(m,n) \sim 0.$$

[C] N **horizontal string fluxes**

They are at the bottom $(m = 0, 1, \cdots, (N-1), n = 0)$ of the lattice.

$$\mathcal{U}(m, 0; \hat{1}) = U(m, 0; \hat{1}) \tag{2.60}$$

and their conjugate electric fields are

$$\mathcal{E}(m, 0; \hat{1}) = \sum_{j=0}^{N} E(m, j; \hat{1}) \tag{2.61}$$

like vertical string fluxes, these horizontal string fluxes are also unphysical or cyclic as their electric fields vanish in physical Hilbert space and they can not appear in the dynamics. This is due to the following:

$$\sum_{j=0}^{N} E(m, j; \hat{1}) = -\sum_{j=0}^{N}\sum_{i=0}^{m-1} \mathcal{G}(i, j) \sim 0. \tag{2.62}$$

Inverse canonical transformations

Now we invert canonical transformations obtained above. The link conjugate pairs are :

[A] Horizontal link fluxes are

$$U(\vec{n}; \hat{1}) = \mathcal{U}(m, 0; \hat{1}) \prod_{j=0}^{n-1} \mathcal{U}^{\dagger}(m, j; \hat{2})\, \mathcal{W}(m, j)\, \mathcal{U}(m+1, j; \hat{2}) \tag{2.63}$$

and their conjugate electric fields are

$$E(m, n; \hat{1}) = \mathcal{E}(m, n) - \mathcal{E}(m, n-1) \tag{2.64}$$

[B] Vertical links fluxes are

$$U(\vec{n}; \hat{2}) = \mathcal{U}(\vec{n}; \hat{2}) \tag{2.65}$$

and their conjugate electric fields are

$$E(m, n; \hat{2}) = \mathcal{E}(m, n; \hat{2}) - \mathcal{E}(m, n) + \mathcal{E}(m-1, n) \tag{2.66}$$

In terms of dual degrees of freedoms, the original Gauss laws take the following forms

$$\mathcal{G}(m, n \neq 0) = \mathcal{E}(m, n; \hat{2}) - \mathcal{E}(m, n-1; \hat{2}) \tag{2.67}$$

At the bottom of the lattice,

$$\mathcal{G}(m, n=0) = \mathcal{E}(m, 0; \hat{1}) - \mathcal{E}(m-1, 0; \hat{1}) + \mathcal{E}(m, 0; \hat{2}) \tag{2.68}$$

Dual dynamics

On the physical Hilbert space, string fluxes decouple and we are only left with physical plaquette degrees of freedom. One can write dual relation in the electric field sector in the following covariant form

$$E(\vec{n}; \hat{i}\,) = -\epsilon_{ij} \nabla_j \mathcal{E}(\vec{n}), \tag{2.69}$$

In (2.69), we have defined $\nabla_i \mathcal{E}(\vec{n}) \equiv \mathcal{E}(\vec{n}) - \mathcal{E}(\vec{n} - \hat{i}\,)$. Now we can write the Hamiltonian in terms of these plaquette fluxes and their conjugate scalar electric potentials:

$$H = g^2 \sum_{\vec{n}, i} (\nabla_i \mathcal{E}(\vec{n}))^2 + \frac{K}{g^2} \sum_{\vec{n}} \left(2 - \mathcal{W}(\vec{n}) - \mathcal{W}^\dagger(\vec{n})\right) \tag{2.70}$$

The dual Hamiltonian is local and simple to interpret. The original 4 link interaction terms which dominate near $g \to 0$ now become non-interacting plaquette or magnetic field terms. On the other hand, free electric field terms now become nearest neighbour interactions of dual scalar potentials. All the interactions now are proportional to g^2 with nearest neighbour dual electric potentials.

2.3.3 U(1) magnetic disorder operator & magnetic vortex

Now we defined the magnetic disorder operator which creates and annihilate the magnetic fluxes at plaquettes. The magnetic flux for any plaquette can be defined as $\mathcal{W}(p) = e^{iB(p)}$. We can interpret $\mathcal{E}(p)$ as magnetic disorder operators which translate U(1) magnetic fields. Such disorder operator is defined as

$$\Sigma_\theta(p) = \exp\{i\,\theta(p)\,\mathcal{E}(p)\} \tag{2.71}$$

In equation (2.71), $\theta(p) \in [0, 2\pi)\ \forall p$ is an arbitrary angle. By construction, this disorder operator is unitary i.e. $\Sigma_\theta(p)\Sigma_\theta^\dagger(p) = \mathcal{I} = \Sigma_\theta^\dagger(p)\Sigma_\theta(p)$. Using the basic canonical commutation relations 2.56, one can establish corresponding order-disorder algebra:

$$\Sigma_\theta(p)\ \mathcal{W}(p)\ \Sigma_\theta^\dagger(p) = e^{i\,\theta(p)}\,\mathcal{W}(p) \tag{2.72}$$

2.3.4 U(1) Order-Disorder algebra

Any Wilson loop $W_{[\mathcal{C}]}$ associated to an arbitrary close curve $\mathcal{C}$ can be expressed as the porduct of plaquette operators $\mathcal{W}(p)$

$$W_{[\mathcal{C}]} = \prod_{p\in\mathcal{C}} (\mathcal{W}(p))^{q_p} \tag{2.73}$$

In the above equation, $q_p = \pm 1$ depending upon the relative orientation of $\mathcal{W}(p)$ and $W_{[\mathcal{C}]}$. Now we can write U(1) order-disorder algebra as

$$W_{[\mathcal{C}]}\ \Sigma_\theta(p) = e^{iq_p\theta_p}\Sigma_\theta(p)W_{[\mathcal{C}]} \tag{2.74}$$

In the above equation, $q_p = +1(-1)$ for clockwise (anti-clockwise) $\mathcal{C}$ and $q_p = 0$, if p does not lie inside $\mathcal{C}$. In order to see that the Wilson loop operator creates electric fluxes along the path $\mathcal{C}$, we construct an electric basis $|r_p\rangle$, the eigenbasis of electric field operators $\mathcal{E}(p)$:

$$\mathcal{E}(p)\ |r_p\rangle = r_p|r_p\rangle, \qquad r_p = 0, 1, 2, \dots \quad \forall p \tag{2.75}$$

Using commutation relations 2.56, it can be shown that the action of the plaquette operators on the above basis will be

$$\mathcal{W}(p)\ |r_p\rangle = |r_p + 1\rangle, \quad \mathcal{W}^\dagger(p)\ |r_p\rangle = |r_p - 1\rangle \qquad r_p = 0, 1, 2, \dots \quad \forall p \tag{2.76}$$

for an arbitrary Wilson loop, one can show that

$$W_{[\mathcal{C}]}\ \prod_{p\in\Lambda}|r_p\rangle = \prod_{p\in\mathcal{C}}|r_p + 1\rangle \prod_{p'\notin\mathcal{C}}|r_{p'}\rangle, \tag{2.77}$$

which implies that the Wilson loop operator creates electric fluxes. Now we consider magnetic field basis $|\omega_p\rangle$, an eigenbasis of plaquette operators $\mathcal{W}(p)$, which is defined as :

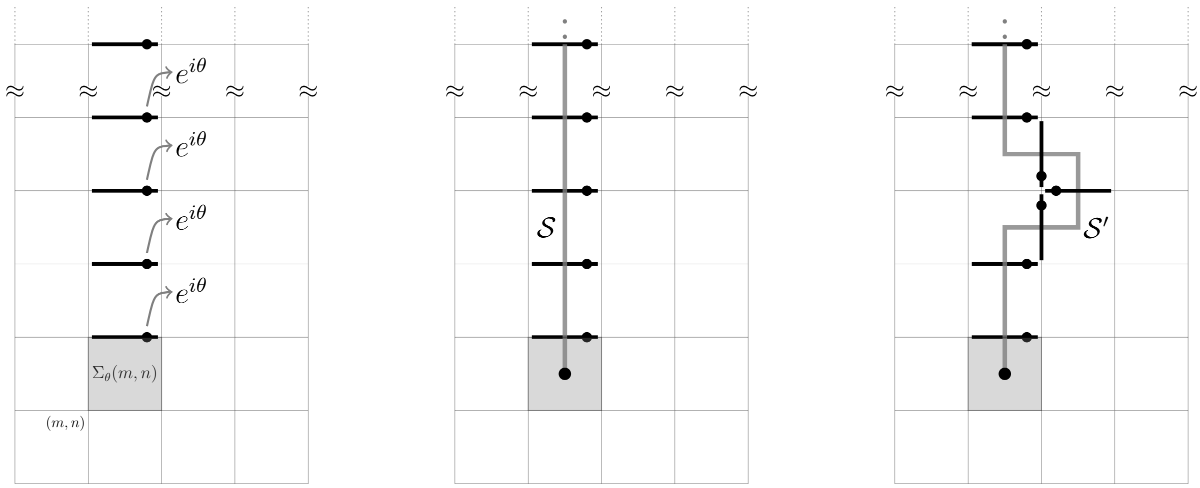
$$\mathcal{W}(p)|\omega_p\rangle \equiv e^{i\omega_p}|\omega_p\rangle \tag{2.78}$$

The above magnetic basis (2.78) is related to electric basis (2.75), through Fourier transform as

$$|\omega_p\rangle = \sum_{r_p=0}^{\infty} e^{ir_p\omega_p}|r_p\rangle, \qquad \forall p \tag{2.79}$$

The action of disorder operators, defined in (2.71) on the magnetic basis is as follows

$$\Sigma_\theta(p)|\omega_p\rangle = e^{i\theta_p\mathcal{E}(p)}\sum_{r_p=0}^{\infty} e^{ir_p\omega_p}|r_p\rangle = \sum_{r_p=0}^{\infty} e^{ir_p\omega_p}e^{i\theta_p\mathcal{E}(p)}|r_p\rangle = \sum_{r_p=0}^{\infty} e^{ir_p\omega_p}e^{i\theta_p r_p}|r_p\rangle$$



(a) Magnetic Vortex (b) Invisible Dirac String (c) Deformation of Dirac String

Figure 2.12: (a) The disorder operator $\Sigma_\theta(m,n)$ defined in (2.81) change all horizontal links $U(m-1, j > n; \hat{1}\,)$, by phase a factor $e^{i\theta}$, (b) Invisible Dirac string, $\mathcal{S}$, (c) Shape of Dirac string can be deformed without affecting the magnetic vortex.

$$= \sum_{r_p=0}^{\infty} e^{ir_p(\omega_p+\theta_p)}|r_p\rangle = |\omega_p + \theta_p\rangle \tag{2.80}$$

This implies that these disorder operators create magnetic fluxes on plaquettes. In terms of original Kogut-Susskind electric fields, these disorder operators have non-local expression but they do create local effects. Using exact duality relation in 2.55, and (2.71) we will have

$$\Sigma_\theta(m,n) = \exp\left\{ i\,\theta \sum_{j=n+1}^{N} E(m,j;\hat{1}) \right\} \tag{2.81}$$

This operator affects all the horizontal link $U(m, j > n; \hat{1})$ by phase factor $e^{i\theta}$, see figure 2.12-(a). Now we will show that the non-local disorder operator in (2.81) is only visible at its endpoint (m,n), therefore creating an invisible Dirac string $\mathcal{S}$, see figure 2.12-(b). From equation (2.72), it is clear that $\Sigma_\theta(m,n)$ creates a magnetic flux at plaquette $U_p(m,n)$. For any plaquette $U_p(m, j > n) = U(m,j;\hat{2})U(m,j+1;\hat{1})U^\dagger(m+1,j;\hat{2})U^\dagger(m,j;\hat{1})$, $\Sigma_\theta(m,n)$ will modify its two horizontal links $U(m,j+1;\hat{1})$ and $U^\dagger(m,j;\hat{1})$ with phase factors $e^{i\theta}$ and $e^{-i\theta}$ respectively, leaving $U_p(m, j > n)$ unchanged. Therefore effects of disorder operators will be visible only at the end. We can also show that the Dirac string can be arbitrarily deformed using Gauss's laws. Any electric field operator $E(m, j > n; \hat{1})$ can be replaced by three electric fields corresponding to the three links emanating from the sites at either side of the link. This replacement will result in the deformation of Dirac string $\mathcal{S}'$, see figure 2.12-c, keeping the physical effects unaffected.

CHAPTER 3

LATTICE GAUGE THEORY

In this chapter, we will briefly review and Kogut-Susskind Hamiltonian formulations (see section 3.1) of lattice gauge theory. Apart from the completeness of the basics of the subject, this will also introduce the notations and conventions which will be used throughout the thesis.

Throughout the thesis, lattice sites and links will be denoted by $\vec{n}$ and $(\vec{n}, \hat{i})$ where $\hat{i}$ is the unit vector along the direction of the link. We often use l and p for the link and the plaquette index for convenience. The plan of this chapter is as follows. We begin with the continuum Yang mils theory and write down the corresponding lattice Lagrangian. Then in order to write the Hamiltonian formalism we define conjugate electric fields and impose canonical commutation relations in section 3.1. In section 3.1.2 we discuss loop formulation of lattice gauge theory and Mandelstam constraints.

3.1 Kogut-Susskind formulation

We now briefly review the lattice formulation of the Hamiltonian pure Yang-Mills theory [56, 128] due to Kogut and Susskind. The Hamiltonian formulation of SU(N) lattice gauge theory is realized on a space lattice with continuous time.

Let's start with the continuum Lagrangian formulation of SU(N) gauge theory which is based on gauge field $A_\mu = A^{\rm a}_\mu T^{\rm a}$, where $T^{\rm a}; {\rm a} = 1, \cdots, {\rm N}^2 - 1$ are the generators of SU(N) algebra in the fundamental representation satisfying ${\rm Tr}\left(T^{\rm a} T^{\rm b}\right) = \frac{1}{2}\delta^{\rm ab}$. The Lagrangian is given by

$$L = \frac{1}{2} \int d^3x \ \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}). \tag{3.1}$$

In the above equation, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ is the field strength and g is the coupling constant. There is an internal (colour) space at each space-time point. Local gauge transformations are basis transformations in this internal space. One can define an operator $U(x_1, x_2)$ which 'parallel transports' a vector in the internal space at x_1 to the internal space at x_2 along a path C.

$$U_C(x_1, x_2) \equiv P_C \ \ e^{ig \int_{x_1}^{x_2} A_\mu(x) dx^\mu}. \tag{3.2}$$

In the above, P_C denotes the path-ordered product along curve C. This is necessary as A_μ at different points do not commute. The parallel transport operator in general depends upon the path C. SU(N) gauge theory can be formulated on a space-time lattice using the parallel transport operator. We consider a hyper cubical space-time lattice with lattice spacing 'a'. We define lattice gauge fields $\bar{A}_\mu(\bar{x})$ at the midpoint $\bar{x}$ of each link where $\bar{A}_\mu(\bar{x})$ is the average value of the continuum gauge field A_μ along the link in the direction μ. A link operator corresponding to a link starting at site $\vec{n}$ along the direction $\hat{i}$ is defined as

$$U(n, \hat{i}) = P e^{ig \int_{x}^{x+\hat{i}} A_\mu(x) dx^\mu} \equiv e^{iag\bar{A}_i(\bar{x})}. \tag{3.3}$$

The link operators have the following properties:

$$U(\vec{n}, \hat{i}\,)\ U^\dagger(\vec{n}, \hat{i}\,) = \mathcal{I}, \quad U^\dagger(\vec{n}, \hat{i}\,)\ U(\vec{n}, \hat{i}\,) = \mathcal{I}, \ \ |U(\vec{n}, \hat{i}\,)| = \mathcal{I}. \tag{3.4}$$

Above, $\mathcal{I}$ is an $\mathrm{N} \times \mathrm{N}$ identity operator and $|U| \equiv \det(U)$. Now consider the product of link operators along a plaquette (say, in the 12 plane) on the lattice. It is given by

$$\begin{aligned} U_{p_{12}} &= e^{iagA_1(x,\, y-a/2)} e^{iagA_2(x+a/2,\, y)} e^{-iagA_1(x,\, y+a/2)} e^{-iagA_2(x-a/2,\, y)} \\ &= e^{iag(A_1(x,\, y)-\nabla_2 A_1(a/2))} e^{iag(A_1(x,\, y)+\nabla_1 A_2(a/2))} e^{-iag(A_1(x,\, y)+\nabla_2 A_1(a/2))} e^{-iag(A_2(x,\, y)-\nabla_1 A_2(a/2))} \\ &= e^{ia^2 gF_{12}+o(a^3)}. \end{aligned} \tag{3.5}$$

Above, we have defined $\nabla_2 A_1(x, y) \equiv \frac{A_1(x,y+a/2)-A_1(x,y)}{(a/2)}$ and $\nabla_1 A_2(x,\, y) \equiv \frac{A_2(x+a/2,y)-A_1(x,y)}{(a/2)}$. This expression reduces to continuum partial derivatives under the limit $a \to 0$. Similarly, link operators along a general plaquette in the $\mu\nu$ plane is given by $U_p = e^{ia^2 gF_{\mu\nu}+o(a^3)}$. Now, $\mathrm{Tr}(U_p)$

is given by

$$\mathrm{Tr}(U_p) = \mathrm{Tr}(1 + ia^2 gF_{\mu\nu} - \frac{a^4g^2}{2!}F^2_{\mu\nu} + \cdots) = \mathrm{Tr}\big(1 - \frac{1}{2}a^4g^2F^2_{\mu\nu} + \cdots\big). \tag{3.6}$$

Above, we have used the fact that $\mathrm{Tr}(F_{\mu\nu}) = \mathrm{Tr}(F^{\mathrm{a}}_{\mu\nu}T^{\mathrm{a}}) = 0$. Since, $\mathrm{Tr}(U_P)$ is not Hermitian, we define the lattice Lagrangian[1] as

$$L = \frac{1}{4a^4g^2}\sum_p \left(2\mathrm{N} - \mathrm{Tr}(U_p + U_p^\dagger)\right). \tag{3.7}$$

Above, the summation is over space-time plaquettes. This Lagrangian reduces to the standard continuum Lagrangian 3.1 under a naive continuum limit. Since we are interested in the Hamiltonian formulation, temporal gauge $A_0 = 0$ is chosen so that the link operators corresponding to the links along the time direction are 1 and continuum limit is performed along the time direction. This leads to the following Lagrangian [74, 128]:

$$L = \sum_{links} \frac{a}{4g^2}\mathrm{Tr}\left(\dot{U}^\dagger(\vec{n},\hat{i}\,)\dot{U}(\vec{n},\hat{i}\,)\right) + \sum_{plaq}\frac{1}{4ag^2}\left(2\mathrm{N} - \mathrm{Tr}(U_p + U_p^\dagger)\right). \tag{3.8}$$

Above, the summation is over spatial links and spatial plaquettes. U_p is the product of link operators around a spatial plaquette. The conjugate momentum of $U_{\alpha\beta}(\vec{n},\hat{i}\,)$ and $U^\dagger_{\alpha\beta}(\vec{n},\hat{i}\,)$ is given by $\Pi_{\alpha\beta}(\vec{n},\hat{i}\,) = \frac{a}{4g^2}(\dot{U}^\dagger_{\alpha\beta}(\vec{n},\hat{i}\,))$ and $\Pi^\dagger_{\alpha\beta}$ respectively. Quantization is achieved by imposing the following canonical quantization condition:

$$\left[U_{\alpha\beta}(\vec{n},\hat{i}\,),\ \Pi_{\gamma\delta}(\vec{m},\hat{j}\,)\right] = i\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ij}\delta_{\vec{m},\vec{n}}. \tag{3.9}$$

The Hamiltonian is

$$H = \sum_{links} \frac{a}{4g^2}\mathrm{Tr}\left(\dot{U}^\dagger(\vec{n},\hat{i}\,)\dot{U}(\vec{n},\hat{i}\,)\right) + \sum_{plaq}\frac{1}{4ag^2}\left(2\mathrm{N} - \mathrm{Tr}(U_p + U_p^\dagger)\right). \tag{3.10}$$

It is convenient to formulate the theory in terms of lie algebra-valued conjugate fields. To this effect, conjugate electric fields are defined [128] as

$$\begin{aligned} E^{\mathrm{a}}_{+}(\vec{n},\hat{i}) &= -i\frac{a}{4g^2}\mathrm{Tr}\left(\dot{U}^\dagger(\vec{n},\hat{i})T^{\mathrm{a}}U(\vec{n},\hat{i}) - h.c\right). \\ E^{\mathrm{a}}_{-}(\vec{n}+\hat{i},\hat{i}) &= -i\frac{a}{4g^2}\mathrm{Tr}\left(\dot{U}(\vec{n},\hat{i})T^{\mathrm{a}}U^\dagger(\vec{n},\hat{i}) - h.c\right). \end{aligned} \tag{3.11}$$

[1]The higher order terms in (3.6) are irrelevant [2, 56] in the renormalization group sense.

The quantization conditions (3.9) imply :

$$[E^{\rm a}_{+}(\vec{n},\hat{i}\,),\ U_{\alpha\beta}(\vec{n},\hat{i}\,)] = -\,(T^{\rm a}U(\vec{n},\hat{i}\,))_{\alpha\beta} \tag{3.12a}$$

$$[E^{\rm a}_{-}(\vec{n}+\hat{i},\hat{i}\,),\ U_{\alpha\beta}(\vec{n},\hat{i}\,)] = \ (U(\vec{n},\hat{i})T^{\rm a})_{\alpha\beta} \tag{3.12b}$$

and

$$[E^{\rm a}_{+}(\vec{n},\hat{i}\,),\ E^{\rm b}_{+}(\vec{n},\hat{i}\,)] = if^{\rm abc}E^{\rm c}_{+}(\vec{n},\hat{i}\,), \qquad [E^{\rm a}_{-}(\vec{n},\hat{i}\,),\ E^{\rm b}_{-}(\vec{n},\hat{i}\,)] = if^{\rm abc}E^{\rm c}_{-}(\vec{n},\hat{i}\,). \tag{3.13}$$

In (3.12b), $f^{\rm abc}$ are the SU(N) structure constants. The commutation relations (3.12b) imply that $E^{\rm a}_{+}(\vec{n},\hat{i}\,)$ and $E^{\rm a}_{-}(\vec{n}+\hat{i},\hat{i}\,)$ rotate the link operator from left and right and therefore called left and right electric field respectively. The location of these left and right electric fields on a link is shown in figure 3.1-a. They are related to each other by the following relation

$$E_{-}(\vec{n}+\hat{i},\hat{i}) = -U^{\dagger}(\vec{n},\hat{i}\,)E_{+}(\vec{n},\hat{i}\,)U(\vec{n},\hat{i}\,). \tag{3.14}$$

In the above equation, $E_{+}(\vec{n},\hat{i}\,) = T^{\rm a}E^{\rm a}_{+}(\vec{n},\hat{i}\,)$. In other words,

$$E^{\rm a}_{-}(\vec{n}+\hat{i},\hat{i}) = -R^{\rm ab}(U^{\dagger}(\vec{n},\hat{i}\,))E^{\rm b}_{+}(\vec{n},\hat{i}\,). \tag{3.15}$$

where :

$$R^{\rm ab}(U(\vec{n},\hat{i}\,)) \equiv 2{\rm Tr}\left(T^{\rm a}U(\vec{n},\hat{i}\,)\,T^{\rm b}U^{\dagger}(\vec{n},\hat{i}\,)\right). \tag{3.16}$$

is a rotation matrix with $R^{T}R = RR^{T} = 1$. Note that the relation (3.14) is consistent with the commutation relations (3.12b) and shows that $E_{-}(\vec{n},\hat{i}\,)$ and $E_{+}(\vec{m},\hat{j}\,)$ mutually commute:

$$[E^{\rm a}_{-}(\vec{n},\hat{i}\,),\ E^{\rm b}_{+}(\vec{m},\hat{j}\,)] = 0. \tag{3.17}$$

From (3.14) it follows that they satisfy the kinematical constraint:

$$\sum_{\rm a=1}^{3} E^{\rm a}_{+}(\vec{n},\hat{i}\,)E^{\rm a}_{+}(\vec{n},\hat{i}\,) = \sum_{\rm a=1}^{3} E^{\rm a}_{-}(\vec{n}+\hat{i},\hat{i}\,)E^{\rm a}_{-}(\vec{n}+\hat{i},\hat{i}\,) \equiv E^{2}(\vec{n},\hat{i}\,), \quad \forall(\vec{n},\hat{i}\,). \tag{3.18}$$

ensuring that their magnitudes are equal. Using eqn. (3.11), it can be shown that $(E_{\pm})^2 \equiv \sum_{\rm a} E^{\rm a}_{\pm}E^{\rm a}_{\pm} = \frac{a^2}{g^4}{\rm Tr}(\dot{U}^{\dagger}\dot{U})$. The Hamiltonian (3.10), when written in terms of the link operators and electric fields, becomes

$$H = \sum_{links} \frac{g^2}{4a}E^{2}(\vec{n},\hat{i}\,) + \sum_{plaq} \frac{1}{4ag^2}\left(2{\rm N} - {\rm Tr}(U_p + U_p^{\dagger})\right)\ . \tag{3.19}$$

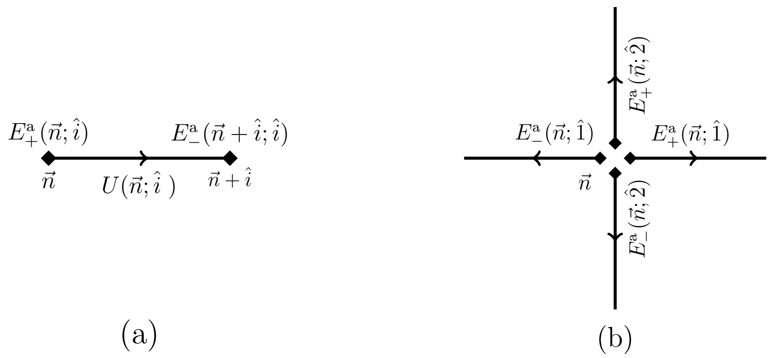


Figure 3.1: Kogut-Susskind link operators $U(\vec{n};\hat{i}\,)$ and their left (right) electric field $E^{\rm a}_{+}(\vec{n};\hat{i}\,)\,(E^{\rm a}_{-}(\vec{n}+\hat{i};\hat{i}\,))$; (a) location of electric fields on the link $(\vec{n};\hat{i}\,)$, (b) location of electric fields around a lattice site $\vec{n}$. The SU(N) Gauss law (3.22) involves all 4 electric fields around site $\vec{n}$.

In (3.20) $\mathrm{Tr}(U_p)$ reduces to $B^{\rm a}B^{\rm a}$ on continuum limit, where $B^{\rm a}$ is along the direction perpendicular to the plaquette p. Now we write the corresponding Hamiltonian on lattice $H_{\rm lattice} \sim 4aH_{\rm continuum}$:

$$H = g^2\sum_{l} E^2(l) + \frac{K}{g^2}\sum_{p}\left(2\mathrm{N} - \mathrm{Tr}(U_p + U_p^{\dagger})\right) \; = H_E + H_B. \tag{3.20}$$

In (3.20) K is a constant. The above Hamiltonian (3.20) reduces to the continuum Hamiltonian under a naive continuum limit.

3.1.1 SU(N) Gauss law constraints

The SU(N) gauge transformations rotates the link operator and the electric fields in the following way:

$$E_{\pm}(\vec{n};\,\hat{i}\,) \rightarrow \Lambda(\vec{n})E_{\pm}(\vec{n},\hat{i}\,)\Lambda^{\dagger}(\vec{n}), \quad U(\vec{n};\,\hat{i}\,) \rightarrow \Lambda(\vec{n})U(\vec{n};\,\hat{i}\,)\Lambda^{\dagger}(\vec{n}+\hat{i}\,). \tag{3.21}$$

The commutation relations (3.12b) along with the gauge transformations (3.21) imply that the generators of SU(N) gauge transformations at any lattice site n are:

$$\mathcal{G}^{\rm a}(\vec{n}) = \sum_{i=1}^{d}\left(E^{\rm a}_{-}(\vec{n};\,\hat{i}\,) + E^{\rm a}_{+}(\vec{n};\,\hat{i}\,)\right), \quad \forall\;\vec{n},\;\mathrm{a}. \tag{3.22}$$

The corresponding Gauss law constraints[2] are

$$\mathcal{G}^{\mathrm{a}}(\vec{n})|\psi_{\mathrm{phys}}\rangle = 0. \tag{3.23}$$

at all lattice sites $\vec{n}$, $|\psi_{\mathrm{phys}}\rangle$ being any state in the physical Hilbert space $\mathcal{H}^{\mathrm{phys}}$ of gauge theory. Hence, the canonical variables $(E_{\pm}(\vec{n};\,\hat{i}\,), U(\vec{n};\,\hat{i}\,))$ are not free and are constrained by (3.22). The Gauss law constraints are illustrated in Figure 3.1(b).

3.1.2 Wilson loops & Mandelstam constraint

In SU(N) lattice gauge theories, one easily obtains a gauge invariant (Wilson) loop basis of the physical Hilbert space $\mathcal{H}^{\mathrm{phys}}$ by applying all possible SU(N) Wilson loop operators on the gauge invariant strong coupling vacuum. However, this simple construction over describes lattice gauge theories. The over-description is because all possible Wilson loop operators are not mutually independent but satisfy constraints known as Mandelstam constraints [2, 49, 50, 52–54, 56, 57, 60, 62, 65, 67, 71, 73–75, 129–137]. They reflect the structure of the gauge group in the form of a set of relations between the loop states of the theory. More precisely, the Mandelstam constraints allow us to express the products of Wilson loops in terms of the sum of the products of a number of loops implying that all loop states in the theory are not mutually independent. These identities were first introduced by Mandelstam for the gauge group O(3) [48, 49]. Extension to GL(N) was achieved by Giles [57].

The Mandelstam constraints are difficult to solve as they involve an arbitrarily large number of non-local loop states of all shapes and sizes. On the other hand, the solutions of the Mandelstam constraints are of significance not only for writing non-Abelian gauge theories without any spurious loop degrees of freedom but also for computing the Hamiltonian spectrum in the weak coupling limit. This is because unlike strong coupling limit ($g^2 \to \infty$), near the weak coupling or continuum ($g^2 \to 0$) limit loop states of arbitrarily large sizes and fluxes become relevant [62, 71, 137]. These constraints become more and more complicated for higher N and higher dimensions. Therefore, they are the major obstacles in loop approaches to gauge theories. In fact, as also mentioned in [60], the loop approach advantages of solving the non-Abelian Gauss law constraints become far less appealing due to the presence of these non-local Mandelstam constraints. In general, a common and widespread belief is that loop formulations of gauge theories, though aesthetically appealing, are seldom practically rewarding due to technical difficulties like the Mandelstam constraints.

In the simplest SU(2) lattice gauge theory case the Mandelstam constraints can be exactly solved in arbitrary dimension using the prepotential approach [61–63, 137–142]. The resulting

[2] Hamiltonian formulation is not equivalent to the Lagrangian formulation unless the Gauss law constraints are imposed. This is because the Lagrangian formulation has Gauss law as an equation of motion while Hamiltonian formulation doesn't and therefore has to be imposed as a constraint on the states.

gauge invariant (loop) basis, also known as the spin network basis, is orthonormal as well as complete. Thus, there are no redundant loop states or SU(2) Mandelstam constraints. The loop basis [62] is characterized by a set of angular momentum quantum numbers. However, the corresponding loop Schrödinger equation involves higher Wigner coefficients (eg; 18-j and 30-j Wigner symbols for 2+1 and 3+1 dimensions respectively) and is extremely complicated to solve. Further, there are numerous (angular momentum) triangular constraints at each lattice site and local Abelian constraints on each link [61–63, 137–142]. All these issues make the prepotential approach less viable for any practical calculation even for the simplest SU(2) case. Also, this approach, when generalized to SU(3) or higher SU(N) lattice gauge theories, further suffers from the problem of multiplicities involved with SU(N) representations [138–140] for $N \geq 3$.

In this thesis, we therefore focus on an alternative approach to loop formulation or duality which involves canonical transformations.

3.2 Canonical transformations, Loops & Duality

In this thesis, we dualize SU(N) Kogut-Susskind Hamiltonian lattice gauge theories [3] in (2+1) dimension in terms of a set of fundamental and physical loop operators and their conjugate loop electric fields [78, 79, 90, 91] using canonical transformations[3]. We start with the standard Kogut-Susskind Hamiltonian formulation of SU(N) lattice gauge theory where the fundamental operators are attached to the links. We then obtain [78, 91] the fundamental loop operators by gluing the standard Kogut-Susskind link operators along the plaquette loops over the entire lattice through a series of iterative canonical transformations. As we have seen in Z_2 and U(1) lattice gauge theories, these fundamental loop description is in terms of the physical magnetic fluxes and their conjugate electric scalar potentials. Thus this loop formulation, obtained through canonical transformations, is also dual to the link formulation by Kogut & Susskind.

In chapter 4 we discuss the SU(N) loop formulation where all loop operators are attached to the origin (see Figure 4.2). Therefore they all rotate together under the gauge transformations at the origin. The canonical transformations also produce a set of SU(N) string flux operators and their conjugate electric fields. The relation between the initial Kogut-Susskind SU(N) link operators and conjugate electric fields and the final physical SU(N) loop, unphysical string operators and their corresponding conjugate electric fields are obtained in a self-consistent manner. In the same chapter, we show that as a consequence of SU(N) Gauss laws, all string degrees of freedom become cyclic or unphysical and decouple, leaving only the physical and mutually independent loop degrees of freedom. As the final loop formulation and the initial Kogut-Susskind link formulation are related through a series of canonical transformations, no

[3]This canonical transformation techniques completely evades the problems of Mandelstam constraints discussed in Section 3.1.2.

extra loop degrees of freedom are generated and hence the problem of Mandelstam constraints is completely evaded for all SU(N) lattice gauge theories. As the plaquette loops are non-local (see Figure 4.2) we are led to non-local loop-loop interactions.

In chapter 5, we use canonical transformations which lead to local plaquette loops (see Figure 5.1). In this formulation, it is possible to make the dual dynamics local through the introduction of auxiliary gauge fields. The local plaquette constraints at each dual site keep the original degrees of freedom intact.

CHAPTER 4

SU(N) LATTICE GAUGE THEORY AND SU(N) SPIN MODEL

In this chapter, we construct SU(N) spin model which is dual to SU(N) lattice gauge theory in $(2+1)$ dimension, see Figure 4.1. This corresponds to the exact SU(N) generalization of the Z_2 Wegner or U(1) duality discussed in Sections 2.2 and 2.3 respectively. As mentioned earlier, dualities in Abelian and non-Abelian lattice gauge theories have been extensively studied in the past [28, 29, 38, 40, 41, 48, 67, 69–77, 94, 98, 143–146]. Most of these studies involved path integral approach and Abelian or SU(2) gauge groups. In some of the above works (4.13) duality transformations involve SU(2) Fourier transformation or character expansion and lead to dual variables which are discrete. They reflect the compactness of the gauge groups and make the topological degrees of freedom manifest [38, 72]. On the other hand, the SU(N) duality transformations discussed in this chapter interchange the role of electric and magnetic degrees of freedom exactly like Z_2 and U(1) cases discussed earlier. We directly motivate the SU(N) results through the Z_2 lattice gauge theory duality discussed in the previous chapter.

Like in Z_2 or U(1) case, the SU(N) spin model is obtained through a series of iterative canonical transformations leading to a formulation of the original SU(N) gauge theory in terms of plaquette loops and their conjugate scalar electric potential [79]. The crucial difference is that unlike Abelian (Z_2 and U(1)) cases where the local plaquette variables are gauge invariant, the local SU(N) plaquette variables transform like adjoint matter. We therefore connect all of them to the origin like shown in Figure 4.2. The SU(N) spins of the dual model are defined as the mutually independent untraced Wilson loops of unit size connected to the origin as shown in Figure 4.2. Therefore this dual formulation can also be interpreted as loop formulation of SU(N) lattice gauge theory. The canonical transformations ensure that the loops are mutual

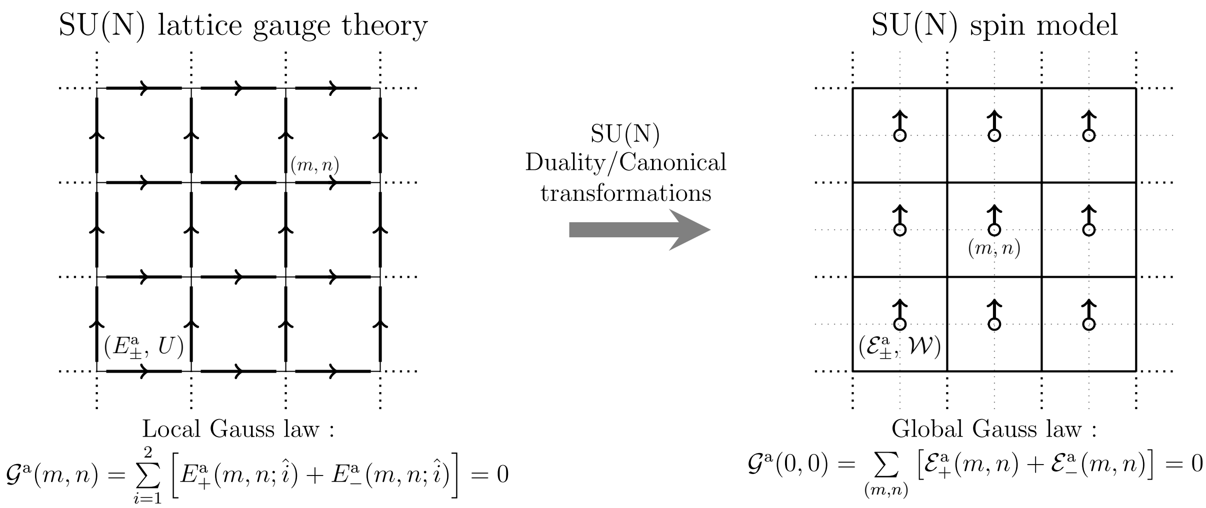


Figure 4.1: Duality between SU(N) lattice gauge theory and an SU(N) spin model. Unlike the corresponding Z_2 duality in Figure 2.3-a,b, global SU(N) Gauss law constraints at the origin remain unsolved due to the non-Abelian nature of the gauge transformations.

independence and free from the Mandelstam constraints [2, 41, 49, 50, 52–54, 57, 67, 71, 73–75, 89, 129–131, 136, 147]. Like Z_2 duality, in this case also the interacting and non-interacting terms get interchanged. The original interacting part of the Hamiltonian, which is a four-gluon interaction in the continuum limit, now becomes the non-interacting magnetic field term. On the other hand, the non-interacting electric field term now describes interactions amongst the dual electric scalar potentials. These interactions are non-local (see (4.13)) with inverted coupling [78]. The SU(N) dual (spin) operators also lead to a new SU(N) disorder operator discussed in chapter 6. For the sake of comparison and convenience, all initial and final dual spin operators involved in Z_2 and SU(N) lattice gauge theories are shown in Table-4.1. All algebraic details of SU(N) canonical transformations can be found in [79]. Note that the Abelian U(1) results discussed in section 2.3 can be easily obtained by ignoring all non-Abelian terms from the duality transformations discussed in this chapter.

Physical sector and SU(N) dual potentials

We consider a finite 2 dimensional lattice with $\mathcal{N} = (N+1)\times(N+1)$ sites, $\mathcal{L} = 2N(N+1)$ links, and $\mathcal{P} = N^2$ plaquettes satisfying $(\mathcal{L} = \mathcal{P} + \mathcal{N} - 1)$. A lattice site is denoted by $\vec{n}$ or (m,n) with $m,n = 0,1,\cdots,N$ and links are denoted by $(m,n;\hat{i}\,)$ where $i = 1,2$. The link conjugate pairs in the Kogut-Susskind formulation are $(E(m,n;\hat{i}\,),\, U(m,n;\hat{i}\,))$. We now define the dual SU(N) spin and SU(N) string operators analogous to the Z_2 spins and strings in (2.26a), (2.26b) and (2.28a), (2.28b) respectively. They are pictorially described in Figure

Z_2 **lattice gauge theory**		**SU(N) lattice gauge theory**	
Gauge Operators	Dual/Spin Operators	Gauge Operators	Dual/Spin Operators
$\{\sigma_1(m,n;\hat{i});\, \sigma_1(m,n;\hat{i})\}$	$\{\mu_1(m,n);\mu_1(m,n)\}$ (Z_2 loops/Z_2 Ising spins) $\{\bar{\sigma}_1(m,n),\bar{\sigma}_1(m,n)\}$ (Frozen Z_2 strings)	$(E(m,n;\hat{i}),\, U(m,n;\hat{i}))$	$(\mathcal{E}_\pm(m,n),\mathcal{W}(m,n))$ (SU(N) loops/ SU(N) spins) $(\mathsf{E}_\pm(m,n),\mathsf{T}(m,n))$ (Frozen SU(N)strings)

Table 4.1: The basic conjugate operators of the original and the dual Z_2, SU(N) gauge theories in $(2+1)$ dimensions.

4.2-a,b respectively. Due to the non-Abelian nature of the electric field and the flux operators, the SU(N) duality relations have additional non-Abelian structures [79]. To begin with, the $\mathcal{N}$ SU(N) Gauss law constraints at $\mathcal{N}$ different lattice sites are all mutually independent. In other words, identities like (2.25) do not exist. As a result, there is a global SU(N) invariance in the SU(N) spin model corresponding to the gauge transformations at the origin. All dual operators transform covariantly under this global SU(N). As shown in Figure 4.2, the SU(N) duality transformations involve parallel transports from the origin to the site of the dual operators. The string flux operator $\mathsf{T}(m,n)$ at a lattice site (m,n) (analogous to $\bar{\sigma}_3(m,n)$ in the Z_2 case) is defined through the path $(0,0)\to(m,0)\to(m,n)$:

$$\mathsf{T}(m,n) = \left(\prod_{m'=0}^{m} U(m',0;\hat{1}) \prod_{n'=0}^{n} U(m,n';\hat{2})\right), \tag{4.1a}$$

$$\mathsf{E}^{\rm a}_{+}(m,n) = \mathcal{G}^{\rm a}(m,n) \approx 0, \tag{4.1b}$$

These strings and their electric fields $\mathsf{E}^{\rm a}_{+}(m,n)$ are shown in Figure 4.2-b and Figure 4.3-b respectively. The relations (6.5a) and (6.5b) are the SU(N) analogues of the Z_2 string relations (2.28a) and (2.28b) respectively.

The dual SU(N) spin and the SU(N) electric scalar potential operators in terms of the original Kogut-Susskind operators are defined [79] as

$$\mathcal{W}(m,n) = \mathsf{T}(m{-}1,n{-}1)\; U_p(m,n)\; \mathsf{T}^\dagger(m{-}1,n{-}1), \tag{4.2a}$$

$$\mathcal{E}^{\rm a}_{+}(m,n) = \sum_{n'=n}^{N} R^{\rm ab}(S(m,n;n'))E^{\rm b}_{-}(m,n';\hat{1}). \tag{4.2b}$$

The above relation (4.2b) is pictorially shown in Figure 4.3. The two operators in (4.2a) and (4.2b) are the non-Abelian extensions of the two Z_2 dual operators defined in (2.26a) and (2.26b) respectively. In (4.2a), (4.2b), the plaquette operator $U_p(m,n)$ and the parallel

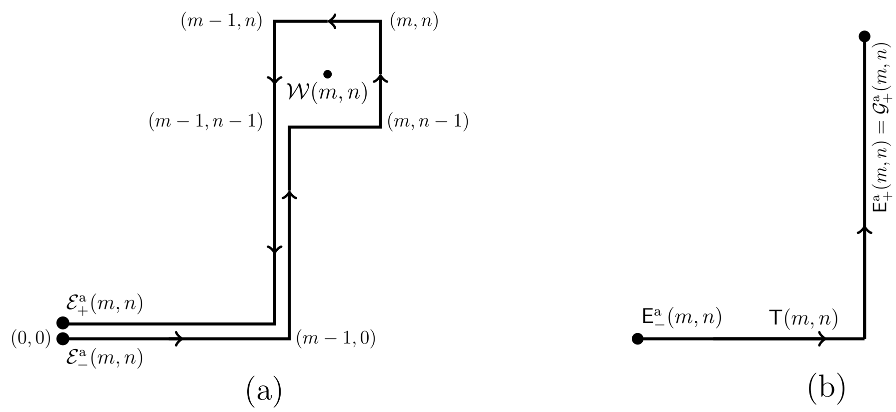


Figure 4.2: The physical SU(N) spin conjugate pairs ($\mathcal{E}_{\pm}$, $\mathcal{W}_{\alpha\beta}$) and the unphysical SU(N) string conjugate pairs ($\mathsf{E}_{\pm}(n)$, $\mathsf{T}(n)$) dual to SU(N) lattice gauge theory are shown in (a) and (b) respectively. Like in Z_2 case in Figure 2.5, we label the SU(N) spin operators by their top right corners and the SU(N) string operators by their endpoints. The strings decouple from the physical Hilbert space as $\mathsf{E}^{\mathrm{a}}_{+}(m,n) \approx 0$ by the Gauss law constraints in $\mathcal{H}^{\mathrm{phys}}$.

transport $S(m,n;n')$ are defined as

$$U_p(m,n) = U(m-1,n-1;\hat{1})\ U(m,n-1;\hat{2})U^{\dagger}(m-1,n;\hat{1})\ U^{\dagger}(m-1,n-1;\hat{2}), \tag{4.3}$$
$$S(m,n;n') \equiv \mathsf{T}(m-1,n)\ U(m-1,n;\hat{1}) \prod_{q=n}^{n'} U(m,q;\hat{2}).$$

The relation (4.2a) defines the SU(N) magnetic field operator as a fundamental operator. The second relation (4.2b) defines SU(N) electric scalar potential $\mathcal{E}^{\mathrm{a}}(m,n)$ which is dual to the original magnetic vector potential. The appearance of the $\mathsf{T}(m,n)$ and $S(m,n;n')$ in (4.2a) add (4.2b) is due to the non-Abelian nature of the operators. These parallel transports from the origin are required to have consistent gauge transformation properties of the SU(N) magnetic fields and the SU(N) electric scalar potentials (see (4.9)).

The dual or loop operators satisfy the expected non-Abelian duality or quantization rules:

$$\left[\mathcal{E}^{\mathrm{a}}_{-}(m,n), \mathcal{W}_{\alpha\beta}(m,n)\right] = -\left(T^{\mathrm{a}}\mathcal{W}(m,n)\right)_{\alpha\beta}, \quad \left[\mathcal{E}^{\mathrm{a}}_{+}(m,n), \mathcal{W}_{\alpha\beta}(m,n)\right] = \left(\mathcal{W}(m,n)T^{\mathrm{a}}\right)_{\alpha\beta}, \tag{4.4a}$$
$$\left[\mathcal{E}^{\mathrm{a}}_{-}(m,n), \mathcal{E}^{\mathrm{b}}_{-}(m,n)\right] = if^{\mathrm{abc}}\mathcal{E}^{\mathrm{c}}_{-}(m,n), \qquad \left[\mathcal{E}^{\mathrm{a}}_{+}(m,n), \mathcal{E}^{\mathrm{b}}_{+}(m,n)\right] = if^{\mathrm{abc}}\mathcal{E}^{\mathrm{c}}_{+}(m,n). \tag{4.4b}$$

Further, the two electric fields are related through parallel transport and commute:

$$\mathcal{E}^{\mathrm{a}}_{-}(m,n) \equiv -R^{\mathrm{ab}}(\mathcal{W}^{\dagger}(m,n))\mathcal{E}^{\mathrm{b}}_{+}(m,n) \qquad \Rightarrow \qquad \left[\mathcal{E}^{\mathrm{a}}_{-}(m,n),\ \mathcal{E}^{\mathrm{b}}_{+}(m,n)\right] = 0. \tag{4.5}$$

The quantization relations (4.4a), (4.4b) and (4.5) are exactly similar to the original quantization rules (3.12b). Thus the electric field operator $E^{\mathrm{a}}(m,n;\hat{i})$ and the magnetic vector potential

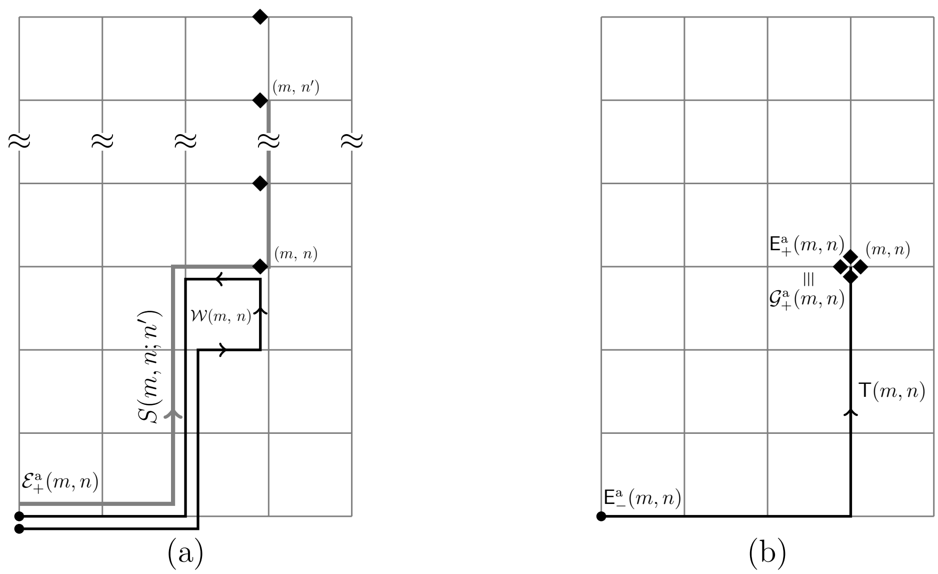


Figure 4.3: The non-local relations in SU(N) duality transformations and the Gauss law constraints.(a) We show the relations (4.2b) expressing $\mathcal{E}^{\mathrm{a}}_{+}(m,n)$ as the sum of $E^{\mathrm{b}}_{-}(m,n';\hat{1})$. The Kogut-Susskind electric fields and the plaquette loop electric fields are denoted by ♦ and • respectively. In (b) we show the SU(N) Gauss law constraints (4.8). The corresponding Z_2 illustrations are in Figure 2.6-a,b,c.

operator $U_{\alpha\beta}(m,n;\hat{i}\,)$ have been replaced by their dual electric scalar potential $\mathcal{E}^{\mathrm{a}}(m,n)$ and the dual magnetic field operator $\mathcal{W}_{\alpha\beta}(m,n)$. This is similar to Z_2 lattice gauge theory duality where $\{\sigma_1(m,n);\sigma_3(m,n)\}$ get replaced by $\{\mu_3(m,n);\mu_1(m,n)\}$. We again emphasize that $\mathcal{E}^{\mathrm{a}}(m,n)$ defines the dual electric scalar potential as it is conjugate to the fundamental magnetic flux operator $\mathcal{W}_{\alpha\beta}(m,n)$.

Unphysical sector and SU(N) string operators

The unphysical sector, representing the gauge degrees of freedom, consists of the string flux operators $\mathsf{T}(m,n)$ in (6.5a) and their conjugate electric fields $\mathsf{E}^{\mathrm{a}}(m,n)$ in (6.5b). They satisfy the canonical quantization relations:

$$\left[\mathsf{E}^{\mathrm{a}}_{-}(m,n),\mathsf{T}_{\alpha\beta}(m,n)\right] = -\left(T^{\mathrm{a}}\mathsf{T}(m,n)\right)_{\alpha\beta}, \quad \left[\mathsf{E}^{\mathrm{a}}_{+}(m,n),\mathsf{T}_{\alpha\beta}(m,n)\right] = \left(\mathsf{T}(m,n)T^{\mathrm{a}}\right)_{\alpha\beta} \tag{4.6a}$$

$$\left[\mathsf{E}^{\mathrm{a}}_{-}(m,n),\ \mathsf{E}^{\mathrm{b}}_{-}(m,n)\right] = if^{\mathrm{abc}}\,\mathsf{E}^{\mathrm{c}}_{-}(m,n), \qquad \left[\mathsf{E}^{\mathrm{a}}_{+}(m,n),\ \mathsf{E}^{\mathrm{b}}_{+}(m,n)\right] = if^{\mathrm{abc}}\,\mathsf{E}^{\mathrm{c}}_{+}(m,n). \tag{4.6b}$$

Again, the operators $\mathsf{E}^{\mathrm{a}}_{+}$ and $\mathsf{E}^{\mathrm{b}}_{-}$ are related through parallel transport and commute amongst themselves:

$$\mathsf{E}^{\mathrm{a}}_{-}(m,n) \equiv -R^{\mathrm{ab}}(\mathsf{T}^{\dagger}(m,n))\mathsf{E}^{\mathrm{b}}_{+}(m,n) \qquad \Rightarrow \quad \left[\mathsf{E}^{\mathrm{a}}_{-}(m,n),\ \mathsf{E}^{\mathrm{b}}_{+}(m,n)\right] = 0. \tag{4.7}$$

The right string electric fields are

$$\mathsf{E}^{\rm a}_{+}(m,n) = \sum_{i=1}^{2}\left[E^{\rm a}_{-}(m,n;\hat{i}) + E^{\rm a}_{+}(m,n;\hat{i})\right] = \mathcal{G}^{\rm a}(m,n) \approx 0, \quad \forall\ (m,n) \neq (0,0). \tag{4.8}$$

Thus as in Z_2 lattice gauge theory, the SU(N) Gauss law constraints freeze all SU(N) string degrees of freedom. This is shown in Figure 4.2-b. As a consequence, all strings (or gauge degrees of freedom) completely decouple from the theory.

The residual Gauss law

Unlike Z_2 lattice gauge theory, the SU(N) Gauss law at the origin is independent of the SU(N) Gauss laws at other sites. In other words, the Abelian identity (2.25) has no non-Abelian analogue. Under this residual global gauge invariance at the origin $\Lambda \equiv \Lambda(0,0)$, all loop operators transform like adjoint matter fields:

$$\mathcal{E}_{\pm}(p) \rightarrow \Lambda\ \mathcal{E}_{\pm}(p)\ \Lambda^{\dagger}, \quad \mathcal{W}(p) \rightarrow \Lambda\ \mathcal{W}(p)\ \Lambda^{\dagger}. \tag{4.9}$$

Above, $\mathcal{E}_{\pm}(p) \equiv \mathcal{E}_{\pm}(m,n), \mathcal{W}(p) \equiv \mathcal{W}(m,n)$ and $\Lambda \equiv \Lambda(0,0)$ is the gauge transformation at the origin. This global invariance at the origin is fixed by the (N^2-1) global SU(N) Gauss laws:

$$\begin{aligned}\mathcal{G}^{\rm a} \equiv \mathcal{G}^{\rm a}(0,0) =& E^{\rm a}_{+}(0,0;\hat{1}) + E^{\rm a}_{-}(0,0;\hat{2}) \\ =& \sum_{m=1}^{\mathsf{N}}\sum_{n=1}^{\mathsf{N}}\left[\underbrace{\mathsf{E}^{\rm a}_{-}(m,n)}_{=0} + \underbrace{\mathcal{E}^{\rm a}_{+}(m,n) + \mathcal{E}^{\rm a}_{-}(m,n)}_{\equiv \mathbb{L}^{\rm a}(m,n)}\right] \\ \equiv& \sum_{m=1}^{\mathsf{N}}\sum_{n=1}^{\mathsf{N}} \mathbb{L}^{\rm a}(m,n) = 0.\end{aligned} \tag{4.10}$$

In (4.10), the total left and right electric flux operators on a plaquette located at $p = (m,n)$ are donated by $\mathbb{L}^{\rm a}(m,n)$ and equations (4.7), (4.8) are used to get $\mathsf{E}^{\rm a}_{-}(m,n) = 0$. It is easy show that the dual SU(N) electric scalar potentials, like Z_2 electric potentials in (2.33), solve the SU(N) Gauss law constraints away from the origin and lead to (4.10) at the origin [79]. The residual global constraints (4.10) can be solved by using the angular momentum or spin network basis [72,73,79,90,137,148,149]. Note that in the Abelian U(1) case there is no residual Gauss law as $\mathbb{L}^{\rm a}(m,n) \rightarrow \mathbb{L}(m,n) \equiv \mathcal{E}_{+}(m,n) + \mathcal{E}_{-}(m,n) \equiv 0$.

Inverse relations

The inverse flux operator relations, analogous to the Z_2 relations (2.30), are

$$\begin{aligned} U(m,n;\hat{1}) =& \mathsf{T}^\dagger(m,n)\ \mathcal{W}(m+1,n)\ \mathcal{W}(m+1,n-1)\cdots\mathcal{W}(m+1,1)\ \mathsf{T}(m+1,n),\\ U(m,n;\hat{2}) =& \ \mathsf{T}(m,n+1)\ \mathsf{T}^\dagger(m,n). \end{aligned} \tag{4.11}$$

The inverse electric field relations, analogous to the Z_2 electric field relations (2.31), are

$$\begin{aligned} E_+^{\rm a}(m,n;\hat{1}) =& R^{\rm ab}(\mathsf{T}(m,n))\Big\{\mathcal{E}_-^{\rm b}(m+1,n+1)+\mathcal{E}_+^{\rm b}(m+1,n)+\delta_{n,0}\sum_{\bar{m}=m+2}^{N}\sum_{\bar{n}=1}^{N}\mathbb{L}^{\rm b}(\bar{m},\bar{n})\Big\},\\ E_+^{\rm a}(m,n;\hat{2}) =& R^{\rm ab}(\mathsf{T}(m,n))\Big\{\mathcal{E}_+^{\rm b}(m+1,n+1)+\\ & \qquad + R^{\rm bc}(W(m,n))\mathcal{E}_-^{\rm c}(m,n+1)+\sum_{\bar{n}=n+2}^{N}\mathbb{L}^{\rm b}(m+1,\bar{n})\Big\}. \end{aligned} \tag{4.12}$$

In the last step in (4.12) we have defined, $R^{\rm ab}(W(m,n)) \equiv R^{\rm ab}\big(\mathcal{W}(m,n)\mathcal{W}(m,n-1)\cdots\mathcal{W}(m,1)\big)$, $\mathbb{L}^{\rm a}(m,n) \equiv \big(\mathcal{E}_-^{\rm a}(m,n)+\mathcal{E}_+^{\rm a}(m,n)\big)$. In the Abelian U(1) case (4.12) involves only the nearest neighbour loop electric fields as there are no color indices $\mathbb{L}^{\rm a} \to \mathbb{L} \equiv 0$ and $R^{\rm ab}(U) \to 1$.

SU(N) dual dynamics

The Hamiltonian of pure SU(N) gauge theory in terms of the dual operators is

$$\begin{aligned} H =& \sum_{m,n\in\Lambda}\Bigg[\frac{g^2}{2}\Bigg\{\Big[\vec{\mathcal{E}}_-(m+1,n+1)+\vec{\mathcal{E}}_+(m+1,n)+\Delta_{XY}(m,n)\Big]^2\\ & \qquad + \Big[\vec{\mathcal{E}}_+(m+1,n+1)+R^{\rm bc}(W(m,n))\vec{\mathcal{E}}^{\rm c}_-(m,n+1)+\Delta_Y(m,n)\Big]^2\Bigg\}\\ & \qquad +\frac{1}{2g^2}\Big(2{\rm N}-({\rm Tr}\ \mathcal{W}(m,n)+h.c)\Big)\Bigg]\\ \equiv& g^2\tilde{H}_E+\frac{1}{g^2}\tilde{H}_B. \end{aligned} \tag{4.13}$$

In (4.13) we have defined [79], $\Delta^{\rm a}_{XY}(m,n) \equiv \delta_{m,0}\sum_{\bar{m}=m+2}^{\rm N}\sum_{\bar{n}=1}^{\rm N}\mathbb{L}^{\rm a}(\bar{m},\bar{n})$ and $\Delta^{\rm a}_{Y}(m,n) \equiv \sum_{\bar{n}=n+2}^{\rm N}\mathbb{L}^{\rm a}(m,\bar{n})$, where $\mathbb{L}(m,n)$ is given in the equation (4.10).

Thus the SU(N) Kogut-Susskind Hamiltonian in its dual description (unlike the Z_2 lattice gauge theory Ising model Hamiltonian) becomes non-local. The non-localities in (4.13) comes from the terms $\mathcal{R}(\mathcal{W}), \Delta^{\rm a}_{XY}(m,n)$ and $\Delta^{\rm a}_{Y}(m,n)$. But, since $\mathcal{R}(\mathcal{W}) = 1+o(g)+o(g^2)+\cdots$

, $\mathbb{L}$ is of order g^n, $(n \geq 1)$ which implies that $\Delta^{\rm a}_{XY}(m,n)$ and $\Delta^{\rm a}_{Y}(m,n)$ are both at least of the order of g. Therefore we expect that in the $g^2 \to 0$ continuum limit, these non-local parts can be ignored to the lowest order at low energies. This leads to a simplified local effective Hamiltonian H_{spin} which may describe pure SU(N) gauge theory at low energies, sufficiently well.

$$H_{spin} = \frac{g^2}{2}\left\{\sum_{p=1}^{\mathcal{P}} 4\vec{\mathcal{E}}^{\,2}(p) + \sum_{<p,p'>} \vec{\mathcal{E}}_-(p)\cdot\vec{\mathcal{E}}_+(p')\right\} + \frac{1}{2g^2}\left\{2N - (Tr\mathcal{W}(p) + h.c)\right\} \tag{4.14}$$

$$\equiv \frac{g^2}{2}\tilde{H}'_E + \frac{1}{2g^2}\tilde{H}_B. \tag{4.15}$$

In (4.15), $< p,p' >$ is used to show the nearest plaquettes. The above simplified SU(N) spin Hamiltonian H_{spin} describes the nearest neighbouring SU(N) spins interacting through their left and right electric fields. All interactions are now contained in the 'electric part' $\tilde{H}'_E$ and the magnetic part $\tilde{H}_B \sim \mathrm{Tr}\mathcal{W}(p)$ is a non-interacting term. As a result, the coupling constant of the dual model is the inverse of that of the original Kogut Susskind model:

$$H^{SU(N)}_{gauge}\big(\tfrac{1}{g^2}\big) \simeq H^{SU(N)}_{spin}\big(g^2\big).$$

We have used $\simeq$ above to state that this equivalence is only within the physical Hilbert space $\mathcal{H}^{\rm phys}$. The above relation is SU(N) analogue of the Z_2 result $H^{Z_2}_{gauge}(\lambda) \simeq H^{Z_2}_{spin}(\lambda^{-1})$ discussed earlier.

Note that the global SU(N) invariance (4.9) of the dual SU(N) spin model is to be fixed by imposing the Gauss law (4.10) at the origin. The degrees of freedom before and after duality match exactly as follows. We have converted the initial $3\mathcal{L}$ Kogut-Susskind link operators into $3\mathcal{P}$ plaquette spin operators and $3(\mathcal{N}-1)$ string operators (see Table 4.1) and $\mathcal{L} = \mathcal{P}+(\mathcal{N}-1)$. There are $\mathcal{N}$ mutually independent Gauss laws in SU(N) (but $(\mathcal{N}-1)$ in Z_2 case) lattice gauge theory. Out of these, $(\mathcal{N}-1)$ freeze the $(\mathcal{N}-1)$ strings. We are thus left with a single Gauss law constraint (4.10) in SU(N) spin model after duality and none in the Z_2 case.

CHAPTER 5

LOCAL SU(N) DUAL DYNAMICS

All duality approaches in the past focused on solving the Abelian or non-Abelian Gauss law constraints to write the electric fields in terms of the dual electric potentials. In Abelian gauge theories such solutions are simple and lead to interesting dynamics [28,38,40,41,94,98,143,144]. However, in non-Abelian cases, the duality attempts have not been very successful. Various solutions of non-Abelian Gauss laws lead to the dual descriptions of dynamics which are involved [29,38,48,49,67,69–77] and often nonlocal [78,79] with difficult physical interpretations. These nonlocal interactions also make them computationally unwieldy.

In this chapter, we illustrate how to evade the above difficulties and transit from the original SU(N) Kogut-Susskind (K-S) electric vector field & magnetic vector potential description (see (3.20) to the (dual) magnetic scalar field & electric scalar potential description (see (5.41)). The dual formulation is also a loop formulation as the dual operators involved are untraced Wilson loops over plaquettes or equivalently the magnetic fields (see Figure 5.1) & their conjugate electric scalar potentials. Under SU(N) gauge transformations they both transform like adjoint matter fields. We find that the nonlocal loop-loop interactions, described by electric scalar potentials, can be made local and converted into minimal coupling by introducing auxiliary SU(N) gauge fields through additional plaquette constraints (see (5.33)). This should be contrasted with the original interactions which are in terms of the magnetic vector potential holonomies around the plaquettes (see (3.20)). This duality between the original plaquette link interactions and the minimal coupling interactions describing loops in $(2 + 1)$ dimension is a novel feature of the present study.

In the previous chapter (4), we have constructed duality transformations which explicitly solved the SU(N) Gauss law constraints at every lattice site. The dual theory in this case was a SU(N) spin model without any gauge degrees of freedom. The above solutions of SU(N) Gauss

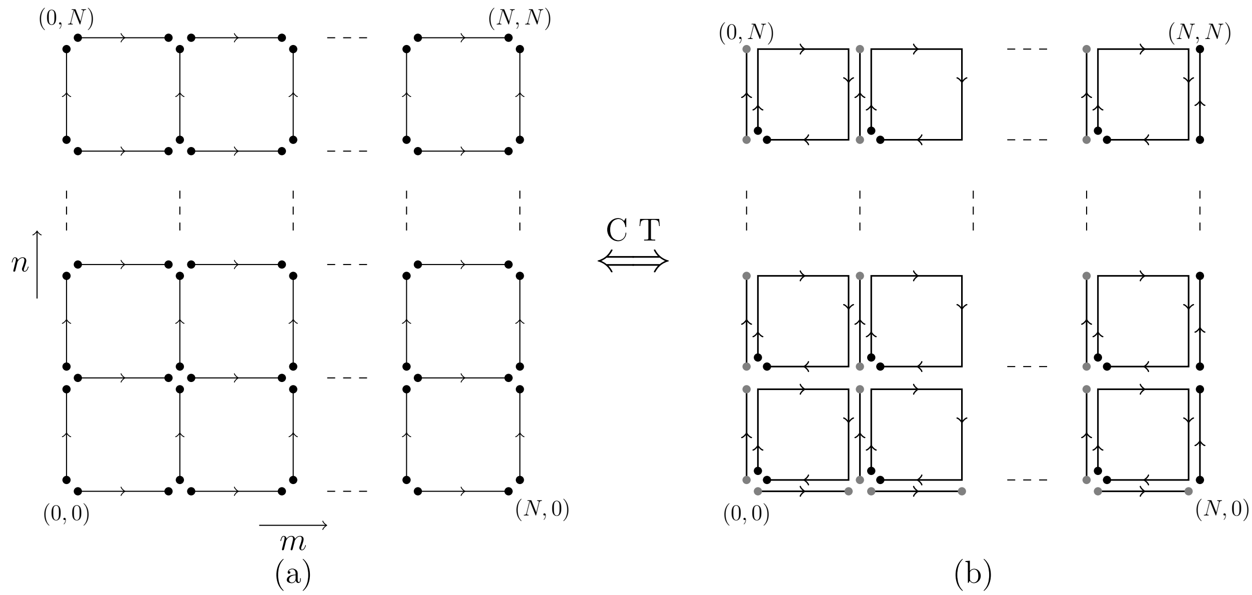


Figure 5.1: (a) Original lattice with $2N(N+1)$ link holonomies and their conjugate electric fields, (b) The final configurations: N^2 plaquette loop holonomies, $N(N+1)$ vertical link holonomies and N horizontal links holonomies at $(m, n = 0)$. The missing N^2 horizontal links at $(m, n > 0)$ have been traded off for N^2 plaquettes through canonical transformations (C T) in (5.20) and (5.21). As expected, the total number of new configurations $(= N^2 + N(N+1) + N = 2N(N+1)$) in (b) match with the total number of initial link configurations in (a).

law constraints are essentially nonlocal relations between the SU(N) Kogut-Susskind electric fields and the dual electric scalar potentials leading to nonlocal dual Hamiltonian [78, 79]. These nonlocality issues in the dual formulations have been recently discussed in the context of quantum simulations in the magnetic basis (see Bauer et. al. in [28, 92, 144, 150–153]). In the present chapter, we take a different route and define the dual SU(N) electric scalar potentials without solving the Gauss law constraints. We construct SU(N) magnetic scalar or plaquette fields & their conjugate electric scalar potentials [48, 49] by making a series of iterative canonical transformations on the original electric vector fields and their conjugate magnetic vector potentials. These canonical transformations are designed to produce local plaquette loop holonomies (physical magnetic fields) by glueing together it's 4 link holonomies (gluons). This framework is pictorially illustrated in Figure 5.1 and Figure 5.3. Following this process we find:

• the Kogut-Susskind non-interacting electric field $g^2\,\vec{E}^2$ terms dualize to loop interaction terms. They are described by minimally coupled electric scalar potentials and the auxiliary SU(N) gauge fields.

• the Kogut-Susskind interacting magnetic field $1/g^2\,\mathrm{Tr}\,(U_{\texttt{plaquette}} + h.c)$ terms dualize to the non-interacting magnetic fields terms. They create and annihilate single plaquette loop.

SU(N) Kogut Susskind Formulation	**Dual SU(N) Formulation**	
[A] (Mixed)	[B] (Physical)	[C] (Unphysical)
Link holonomy: $U_{\alpha\beta}(\vec{n};\hat{i})$	Plaquette holonomy: $\mathcal{W}_{\alpha\beta}(\vec{n})$	String holonomy: $\mathcal{U}(\vec{n};\hat{i})$
Link electric field: $E^{\mathrm{a}}_{\pm}(\vec{n};\hat{i})$	Plaquette potential: $\mathcal{E}^{\mathrm{a}}_{\pm}(\vec{n})$	String electric field: $\mathcal{E}^{\mathrm{a}}_{\pm}(\vec{n};\hat{i})$
$[E^{\mathrm{a}}_{+}(\vec{n};\hat{i}),\ U_{\alpha\beta}(\vec{n};\hat{i})] = -(T^{\mathrm{a}}U(\vec{n};\hat{i}))_{\alpha\beta}$	$[\mathcal{E}^{\mathrm{a}}_{+}(\vec{n}),\ \mathcal{W}_{\alpha\beta}(\vec{n})] = -(T^{\mathrm{a}}\mathcal{W}(\vec{n}))_{\alpha\beta}$	$[\mathcal{E}^{\mathrm{a}}_{+}(\vec{n};\hat{i}),\ \mathcal{U}_{\alpha\beta}(\vec{n};\hat{i})] = -(T^{\mathrm{a}}\mathcal{U}(\vec{n};\hat{i}))_{\alpha\beta}$

Table 5.1: The kinematical degrees of freedom before and after duality transformations. Under SU(N) gauge transformations, the conjugate pairs $(\mathcal{E}^{\mathrm{a}}_{+}(\vec{n}),\ \mathcal{W}_{\alpha\beta}(\vec{n}))$ in [B], defined at lattice site $(\vec{n})$, transform like SU(N) adjoint scalar matter fields. They describe the SU(N) magnetic fields and their conjugate electric scalar potentials respectively. The fields in the last column are the auxiliary SU(N) gauge fields defined on links. They are introduced with additional plaquette constraints (5.33) to obtain minimally coupled local dual theory. This table also explains the notations used in this thesis.

Thus under duality the roles of interacting & non-interacting terms get interchanged resulting in the inversion of coupling constant ($g^2 \to 1/g^2$) as expected.

This chapter is organized as follows: In Section 5.1 we discuss the canonical transformations which take us from link description to the plaquette loop description by joining the 4 links of every plaquette. To make the presentation simple, we first discuss how to join two link holonomies by making a single canonical transformation. In Section 5.1.1 we iterate this step on a simple 2×2 plaquette lattice and define 4 plaquette loop holonomies (magnetic fields) and their conjugate electric scalar potentials. In Section 5.1.2 we directly generalize these results to $N \times N$ plaquette lattice. All technical issues and details involved in performing canonical transformations are worked out in Appendix B. In Section 5.2 we discuss the dual loop dynamics in $(2+1)$ dimensions in terms of magnetic scalar fields & their conjugate electric scalar potentials. The nonlocality problem and it's resolution are also discussed. In this section we also compare our SU(N) duality results with the U(1) lattice gauge theory duality results. This simple comparison provides better understanding of the non-Abelian duality relations between electric fields and the electric scalar potentials.

The notations used are as follows: The lattice sites and links will be denoted by $\vec{n} = (m, n)$ and $(\vec{n};\ \hat{i}\,)$ respectively with $m, n = 0, 1, 2, \cdots, N$ and $i = 1, 2$. We use roman & calligraphic fonts to denote the SU(N) conjugate field operators in the Kogut-Susskind or electric (before duality) & magnetic (after duality) descriptions respectively. This is clearly illustrated in Table 5.1.

5.1 Canonical Transformations: Links to Loops & Strings

In this section, using canonical transformations, we transit from the Kogut-Susskind link electric field representation to it's dual plaquette magnetic field representation in SU(N) lattice gauge theory to write H in (3.20) in it's dual form (5.41). This duality is achieved by canonical gluing the 4 links a round every plaquette on the $N \times N$ lattice and defining the corresponding electric fields. This is pictorially shown in Figure 5.2 and Figure 5.3. As the procedure is iterative, we start with gluing two link holonomies and define their electric fields. We then generalize this canonical transformation procedure to 2×2 plaquette lattice (see section 5.1.1) and then to $N \times N$ plaquette lattice (see section 5.1.2) respectively. In what follows, we will construct only left (right) plaquette & string electric fields through canonical transformations. Their right (left) electric fields can then be easily obtained using the parallel transport relations (3.14) with $U(\vec{n};\hat{i})$ replaced by the corresponding plaquette & string holonomies respectively. We use calligraphic symbols to denote the new field operators obtained after every canonical transformation. We consider two conjugate pairs: $\big(E^{\rm a}_{\pm}(1),\, U_{\alpha\beta}(1)\big)$ and $\big(E^{\rm a}_{\pm}(2),\, U_{\alpha\beta}(2)\big)$. We join them together to define $\big(\mathcal{E}^{\rm a}_{\pm}(12),\, \mathcal{U}_{\alpha\beta}(12)\big)$ and $\big(\mathcal{E}^{\rm a}_{\pm}(2),\, \mathcal{U}_{\alpha\beta}(2)\big)$. This is shown in Figure 5.2. As is clear from this Figure the canonical relations between new and old conjugate operators are

$$\begin{aligned} \mathcal{U}_{\alpha\beta}(12) &\equiv \big(U(1)\, U(2)\big)_{\alpha\beta}, \quad & \mathcal{E}^{\rm a}_{+}(12) &= E^{\rm a}_{+}(1), \\ \mathcal{U}_{\alpha\beta}(2) &\equiv U_{\alpha\beta}(2), & \mathcal{E}^{\rm a}_{+}(2) &= E^{\rm a}_{-}(1) + E^{\rm a}_{+}(2). \end{aligned} \tag{5.1}$$

The transformations (5.1) are canonical as the two new conjugate pairs $\big(\mathcal{E}^{\rm a}_{\pm}(12),\, \mathcal{U}_{\alpha\beta}(12)\big)$ and $\big(\mathcal{E}^{\rm a}_{\pm}(2),\, \mathcal{U}_{\alpha\beta}(2)\big)$ follow the standard canonical commutation relations:

$$[\mathcal{E}^{\rm a}_{+}(12),\, \mathcal{U}_{\alpha\beta}(12)] = -\left(T^{\rm a}\mathcal{U}(12)\right)_{\alpha\beta} \tag{5.2a}$$

$$[\mathcal{E}^{\rm a}_{+}(2),\, \mathcal{U}_{\alpha\beta}(2)] = -\left(T^{\rm a}\mathcal{U}(2)\right)_{\alpha\beta} \tag{5.2b}$$

Note that the two new holonomies $\mathcal{U}_{\alpha\beta}(12)$ and $\mathcal{U}_{\alpha\beta}(2)$ trivially commute with each other. We have added $E^{\rm a}_{-}(1)$ in defining $\mathcal{E}^{\rm a}_{+}(2)$ in (5.1) so that

$$[\mathcal{E}^{\rm a}_{+}(2),\, \mathcal{U}_{\alpha\beta}(12)] = 0, \qquad [\mathcal{E}^{\rm a}_{+}(12),\, \mathcal{U}_{\alpha\beta}(2)] \equiv 0. \tag{5.3}$$

The two new conjugate pairs commute with each other and are mutually independent. They are therefore on the same footing as the original two pairs $\big(E^{\rm a}_{\pm}(1),\, U_{\alpha\beta}(1)\big)$ and $\big(E^{\rm a}_{\pm}(2),\, U_{\alpha\beta}(2)\big)$. We further note that the electric field $E_{\pm}(1)$ of the link holonomy $U_{\alpha\beta}(1)$, which is canonically converted into $\mathcal{U}_{\alpha\beta}(12)$ appears in both the final electric fields. This aspect is clearly shown in Figure 5.2. This simple fact will lead to the nonlocal duality relations (see (5.6) and (5.21)) which are obtained after a series of canonical transformations. This, in turn, will lead to nonlocal dual or loop dynamics (see (5.31)). As mentioned earlier, having defined the left

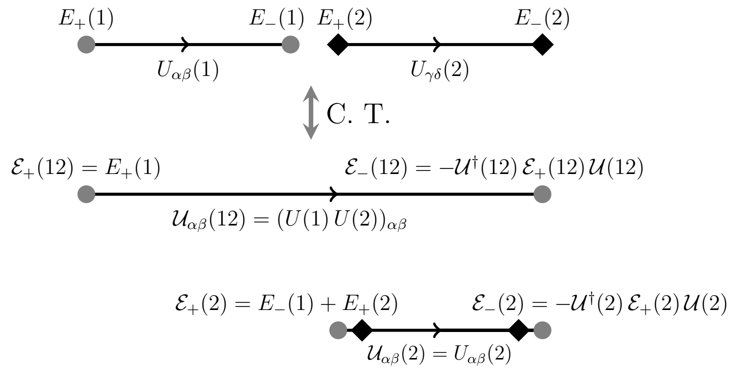


Figure 5.2: Gluing two SU(N) holonomies using canonical transformations. From two Kogut-Susskind links, we get two new mutually independent holonomies.

electric fields in (5.1), the right electric fields get fixed by the parallel transport along the new links

$$\mathcal{E}_+^{\rm a}(12) \;= -R^{\rm ab}(\mathcal{U}(12))\;\mathcal{E}_-^{\rm b}(12), \qquad \mathcal{E}_+^{\rm a}(2) \;\;= -R^{\rm ab}(\mathcal{U}(2))\;\mathcal{E}_-^{\rm b}(2). \tag{5.4}$$

The new left and right electric fields also satisfy the SU(N) Lie algebra, they commute with each other and their magnitudes are equal. In summary, in this section, we have converted the shorter flux line $U_{\alpha\beta}(1)$ into a longer flux line $\mathcal{U}_{\alpha\beta}(12)$ using (5.1). This simple canonical transformation will now be iterated over the entire lattice to convert all horizontal links into local plaquettes starting from the top. This in turn will define the holonomy around a plaquette or the magnetic fields as the fundamental variables in the dual theory [1]. We first generalize the canonical transformations (5.1) to 2×2 plaquette lattice in section 5.1.1 and then discuss the general $N \times N$ plaquette case in section 5.1.2.

5.1.1 (2×2) plaquette lattice

This simple case is illustrated in Figure 5.3 and in Table 5.1. The initial 12 Kogut-Susskind link conjugate pairs $(E(m,n,\hat{i}),\, U(m,n;\hat{i}))$ are shown in figure 5.3-a. The final 4 (physical) plaquette conjugate pairs $(\mathcal{E}(m,n),\, \mathcal{W}(m,n))$ and the remaining 8 (unphysical) string conjugate pairs $(\mathcal{E}(m,n;\hat{i}\,), \mathcal{U}(m,n;\hat{i}\,))$ are shown in figure 5.3-b. As the figure shows, we have converted the 4 horizontal link holonomies and their electric fields at $(m = 0,1, n = 1,2)$ into the 4 plaquette holonomies and their electric fields. The 12 canonical transformations leading to the configurations in Figure 5.3-b from 5.3-a are systematically worked out in Appendix B. In the next section, the end results of the above canonical transformations are written down. They

[1]Note that each plaquette construction requires 3 canonical transformations. Therefore we need to make $3N^2$ canonical transformations to construct the N^2 new conjugate operators defined on plaquettes.

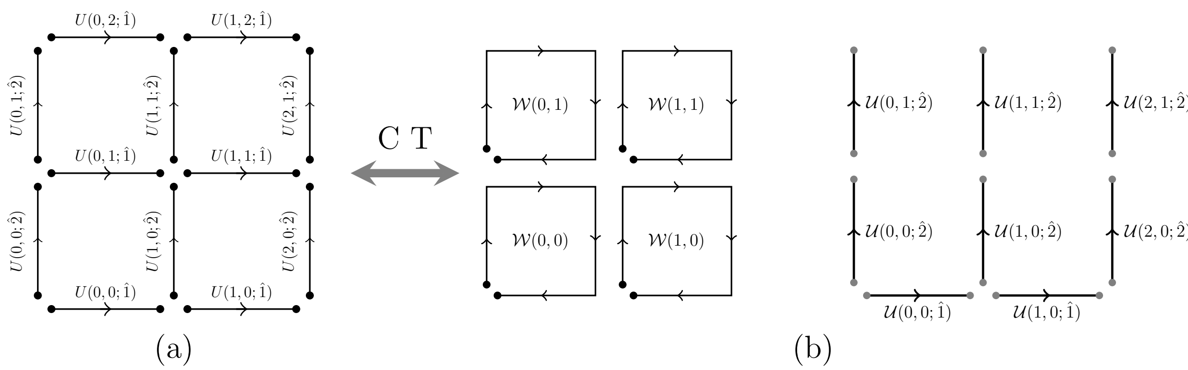


Figure 5.3: A simple 2 × 2 plaquette lattice. (a) The Kogut-Susskind description in terms of 12 link holonomies and their left & right electric fields shown by •, (b) Dual lattice with 4 physical plaquette holonomies $\mathcal{W}(m,n); m,n = 0,1$, 6 unphysical vertical strings $\mathcal{U}(m,n;\hat{2}); m = 0,1,2,\ n = 0,1$ and 2 unphysical horizontal strings $\mathcal{U}(m,0;\hat{1}); m = 0,1$ at the bottom of the lattice. The electric fields of plaquettes are shown by • whereas electric fields of unphysical strings are shown in •. All unphysical strings can be removed by gauge transformations at $(m \neq 0, n \neq 0)$.

have exact duality interpretation.

Plaquette, Strings & Duality

We first describe the new plaquette sector. The 4 plaquette fluxes shown in Figure 5.3-b are

$$\mathcal{W}(m,n) = U(m,n;\hat{2})\, U(m,n+1;\hat{1})\, U^{\dagger}(m+1,n;\hat{2})\, U^{\dagger}(m,n;\hat{1}). \tag{5.5}$$

In (5.5) $m,n = 0,1$. Their conjugate electric fields are (see Appendix B)

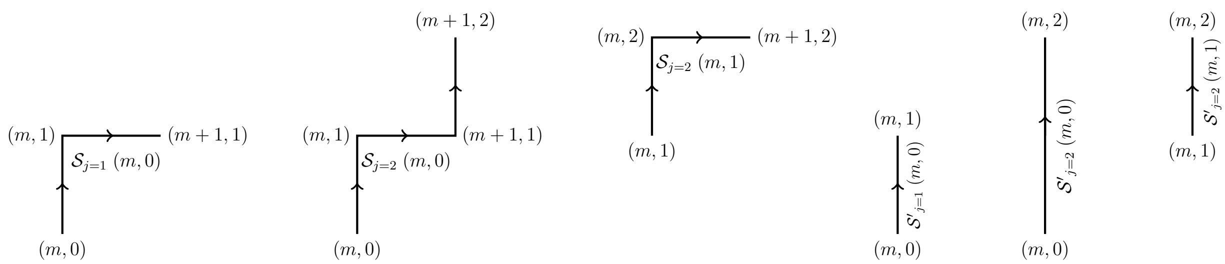
$$\mathcal{E}_{+}(m,n) = -\sum_{j=n+1}^{2} \mathcal{S}_j(m,n) E_{+}(m+1,j;\hat{1}) \mathcal{S}_j^{-1}(m,n). \tag{5.6}$$

The physical interpretation of the nonlocal operators $\mathcal{S}_j(m,n)$ is simple. They implement the parallel transports from the location of the link electric fields at $(m+1, j(= n+1,\cdots,N))$ to the location of the plaquette electric field at (m,n) in (5.6) making the above canonical relations covariant under SU(N) gauge transformations (see Figure 5.4-a,b,c). They are defined as (see Appendix B)

$$\mathcal{S}_{j=1}(m,0) = U(m,0;\hat{2})\, U(m,1;\hat{1}), \tag{5.7a}$$

$$\mathcal{S}_{j=2}(m,0) = U(m,0;\hat{2})\, U(m,1;\hat{1})\, U(m+1,1;\hat{2}), \tag{5.7b}$$

$$\mathcal{S}_{j=2}(m,1) = U(m,1;\hat{2})\, U(m,2;\hat{1}). \tag{5.7c}$$



(a) (b) (c) (d) (e) (f)

Figure 5.4: Non-local parallel transports $\mathcal{S}$ and $\mathcal{S}'$ required for defining plaquette and string electric field operators in (5.7a), (5.7b), (5.7c) and (5.13a), (5.13b), (5.13c), respectively.

The parallel transports in (5.7a), (5.7b) and (5.7c) at $m = 0, 1$ are shown in Figures 5-a,b,c respectively. See equations (B.23), (B.43), (B.29), (B.49), (B.7) and (B.38) in Appendix B. As a consequence of canonical transformations the plaquette pairs $(\mathcal{E}_\pm(m,n), \mathcal{W}(m,n))$ are conjugate and satisfy the standard canonical commutation relations:

$$[\mathcal{E}^{\rm a}_{+}(\vec{n}),\ \mathcal{W}_{\alpha\beta}(\vec{n})] = -\left(T^{\rm a}\mathcal{W}(\vec{n})\right)_{\alpha\beta}, \qquad [\mathcal{E}^{\rm a}_{-}(\vec{n}),\ \mathcal{W}_{\alpha\beta}(\vec{n})] = \left(\mathcal{W}(\vec{n})T^{\rm a}\right)_{\alpha\beta}. \tag{5.8}$$

The plaquette electric fields satisfy SU(N) Lie algebra

$$[\mathcal{E}^{\rm a}_{\pm}(\vec{n}),\ \mathcal{E}^{\rm b}_{\pm}(\vec{n})] \;=\; i f^{\rm abc}\mathcal{E}^{\rm c}_{\pm}(\vec{n}) \tag{5.9}$$

The canonical commutation relation amongst the new plaquette fields (5.8), (5.9) are the dual version of the standard Kogut-Susskind commutation relations (3.12b), (3.13) respectively. Note that while Kogut-Susskind relations (3.12b) involve electric fields $E(\vec{n};\hat{i}))$ and it's conjugate magnetic vector potentials in $U_{\alpha\beta}(\vec{n};\hat{i})$, the dual commutation relations (5.8) involve magnetic scalar fields in $\mathcal{W}(m,n)$ and their conjugate electric scalar potentials $\mathcal{E}(m,n)$. Therefore, the canonical transformations (5.5) and (5.6) (see Appendix B for details) can also be interpreted as the exact SU(N) duality transformations. After duality, the fundamental conjugate pairs describing the dynamics (see Section 5.2) are the magnetic scalar fields and their conjugate electric scalar potentials $(\mathcal{W}(m,n), \mathcal{E}(m,n))$. Under SU(N) gauge transformations (3.22) they transform as adjoint scalar matter fields

$$\mathcal{W}(\vec{n}) \to \Lambda(\vec{n})\mathcal{W}(\vec{n})\Lambda^{\dagger}(\vec{n}), \qquad \mathcal{E}_{\pm}(\vec{n}) \;\to \Lambda(\vec{n})\mathcal{E}_{\pm}(\vec{n})\Lambda^{\dagger}(\vec{n}) \tag{5.10}$$

The electric fields $E(\vec{n};\hat{i}\,)$ are no longer fundamental and can be derived from the electric scalar potentials $\mathcal{E}^{\rm a}(m,n)$ as discussed in the next section (see (5.18a), (5.18b), (5.18c)).

We now describe the remaining unphysical string sector shown in Figure 5.3-b. As the iterative canonical transformations preserve the total number of canonical degrees of freedom at every step, these 8 strings represent the left over degrees of freedom. They are completely unphysical as they can not be part of any gauge invariant operator. This is also clear from the Figure 5.3-b. The 2 horizontal and 6 vertical string holonomies are

$$\mathcal{U}(m,0;\hat{1}) = U(m,0;\hat{1}); \quad m=0,1. \tag{5.11a}$$

$$\mathcal{U}(m,n;\hat{2}) = U(m,n;\hat{2}); \quad m=0,1,2,\ n=0,1, \tag{5.11b}$$

and their conjugate electric fields are

$$\mathcal{E}_+(m,0;\hat{1}) = \ E_+(m,0;\hat{1}) - \sum_{j=1}^{2} \mathcal{S}_j(m,0)\ E_-(m+1,j;\hat{1})\ \mathcal{S}_j^{-1}(m,0), \tag{5.12a}$$

$$\mathcal{E}_+(m,n;\hat{2}) = \ E_+(m,n;\hat{2}) - \sum_{j=n+1}^{2} \mathcal{S}'_j(m,n)\ E_-(m,j;\hat{1})\ \mathcal{S}'^{-1}_j(m,n) + \sum_{j=n+1}^{2} \mathcal{S}_j(m,n)\ E_-(m+1,j;\hat{1})\mathcal{S}_j^{-1}(m,n). \tag{5.12b}$$

Again like the parallel transport $\mathcal{S}_j(m,n);(m,n=0,1)$ the parallel transports $\mathcal{S}'_j(m,n)$ in (5.12b) are required to define the new string electric fields (see Figure 5.4). They are defined as

$$\mathcal{S}'_1(m,0) = U(m,0;\hat{2}), \tag{5.13a}$$

$$\mathcal{S}'_2(m,0) = U(m,0;\hat{2})\, U(m,1;\hat{2}), \tag{5.13b}$$

$$\mathcal{S}'_2(m,1) = U(m,1;\hat{2}). \tag{5.13c}$$

The parallel transports $\mathcal{S}'_j(m,n)$ in (5.13a), (5.13b) and (5.13c) are shown in Figure 5.4-d,e,f respectively. For more details, see equations (B.24), (B.44), (B.50), (B.14) and (B.39) in Appendix B. The 8-string conjugate pairs also satisfy the standard canonical commutations relations:

$$[\mathcal{E}_+^{\mathrm{a}}(m,n;\hat{i}\,),\ \mathcal{U}_{\alpha\beta}(m,n;\hat{i}\,)] = -\Big(T^{\mathrm{a}}\mathcal{U}(m,n;\hat{i}\,)\Big)_{\alpha\beta} \tag{5.14}$$

In (5.14) if $\hat{i} = \hat{1}$ then $n=0$ as horizontal strings exist only at the bottom. These canonical relations are systematically derived in Appendix B. It is easy to check that the 12 new conjugate pairs commute with each other and therefore are completely independent and thus ensuring no mismatch between the initial (link) and the final (loop) degrees of freedom. The new string conjugate pairs transform as SU(N) gauge fields

$$\mathcal{U}(\vec{n};\hat{i}\,) \rightarrow \Lambda(\vec{n})\mathcal{U}(\vec{n};\hat{i}\,)\,\Lambda^\dagger(\vec{n}+\hat{i}\,), \qquad \mathcal{E}_\pm(\vec{n};\hat{i}\,) \ \rightarrow \Lambda(\vec{n})\,\mathcal{E}_\pm(\vec{n};\hat{i}\,)\,\Lambda^\dagger(\vec{n}) \tag{5.15}$$

In (5.15) the conjugate pairs $(\mathcal{E}_\pm(\vec{n};\hat{i}=1),\ \mathcal{U}(\vec{n};\hat{i}=1))$ exist only when $\vec{n}=(m,0)$. Note that this asymmetry is due to the special choice of canonical transformations in Appendix B which convert all horizontal links at $n>0$ into plaquettes [2]. Further, the unphysical string or link fields $\mathcal{U}(\vec{n};\hat{i})$ can be completely gauged away using (5.15). However, our aim is to keep them to obtain local dual dynamics in section 5.2.

Inverse relations

The canonical relations (5.5), (5.11a) and (5.11b) can be easily inverted to write Kogut-Susskind fields in terms of the new plaquette, string fields. The original link holonomies are

$$U(\vec{n};\hat{1}\,) = \mathcal{L}\,(\vec{n})\mathcal{U}(m,0;\hat{1}\,)\,\mathcal{R}(\vec{n}+\hat{1}\,),\quad n=1,2 \tag{5.16a}$$

$$U(m,0;\hat{1}\,) = \mathcal{U}(m,0;\hat{1}\,),\qquad m=0,1 \tag{5.16b}$$

$$U(m,n;\hat{2}\,) = \mathcal{U}(m,n;\hat{2}\,),\qquad m=0,1,2. \tag{5.16c}$$

In (5.16a), the parallel transports on the left and right sides are

$$\mathcal{L}(\vec{n}) \equiv \left[\prod_{j=1}^{n}\mathcal{U}^\dagger(m,n-j;\hat{2}\,)\mathcal{W}(m,n-j)\right] \tag{5.17a}$$

$$\mathcal{R}(\vec{n}+\hat{1}\,) \equiv \left[\prod_{k=0}^{n-1}\mathcal{U}(m+1,k;\hat{2}\,)\right] \tag{5.17b}$$

Note that the nontrivial relations (5.16a) and (5.17a), (5.17b) involving parallel transports $\mathcal{L}$ and $\mathcal{R}$ are again simple consequence of SU(N) gauge covariance under (5.10) and (5.15). The conjugate Kogut Susskind electric fields are

$$E_+(m,0;\hat{1}) = \mathcal{E}_-(m,0)+\mathcal{E}_+(m,0;\hat{1}) \tag{5.18a}$$

$$E_+(\vec{n};\hat{1}) = \mathcal{E}_-(\vec{n})+\mathcal{U}^\dagger(\vec{n}-\hat{2};\hat{2})\mathcal{E}_+(\vec{n}-\hat{2})\mathcal{U}(\vec{n}-\hat{2};\hat{2}) \tag{5.18b}$$

$$E_+(\vec{n};\hat{2}) = \mathcal{E}_+(\vec{n})+\mathsf{S}^{-1}(\vec{n}-\hat{1};\hat{1})\mathcal{E}_-(\vec{n}-\hat{1})\mathsf{S}(\vec{n}-\hat{1};\hat{1})+\mathcal{E}_+(\vec{n};\hat{2}) \tag{5.18c}$$

In (5.18c) nonlocal parallel transport are

$$\mathsf{S}(m,n;\hat{1}) = \mathcal{L}\,\mathcal{U}(m,0;\hat{1})\,\mathcal{R} \tag{5.19}$$

These relations are derived in Appendix B but they are easy to understand and can be written down just by looking at the final 12 $\mathcal{W}$, $\mathcal{U}$ configurations in Figure 5.3-b. We note that the Kogut Susskind electric fields $E_\pm(m,n,\hat{i}\,)$ rotates all those new configurations in Figure 5.3-b

[2] In section 5.2.1 we will reintroduce the missing horizontal links through plaquette constraints (5.33) to get symmetry as well as locality.

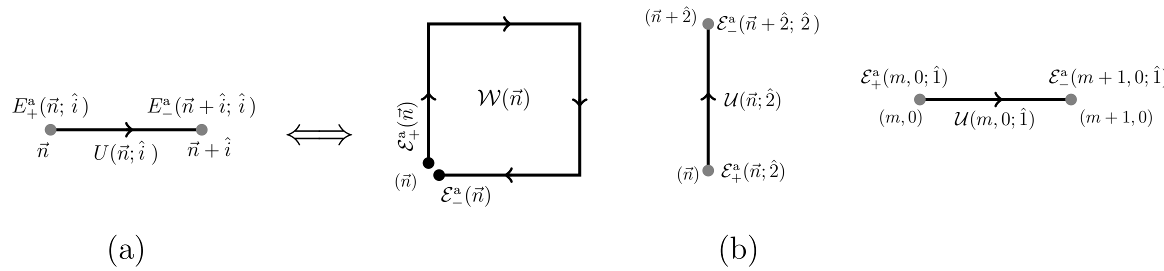


Figure 5.5: Original and final three types of holonomies and their electric fields obtained after canonical transformations (5.1): (a) Kogut-Susskind Link operators, (b)N^2 plaquettes, N^2+N vertical strings & N horizontal string at the bottom of the lattice. The two types of electric fields $E_\pm(\vec{n};\hat{i})$, $\mathcal{E}_\pm(\vec{n})$, $\mathcal{E}_\pm(\vec{n};\hat{i})$ and their locations are shown.

which share the link holonomy $U(m,n,\hat{i}\,)$. Therefore, $E_\pm(m,n,\hat{i}\,)$ is a sum of all these $\mathcal{W}$, $\mathcal{U}$ electric fields parallel transported appropriately to maintain the SU(N) gauge covariance. As an example, if we want to write the Kogut-Susskind left electric field $E_+(m,n;\hat{1})$, we have to identify all the dual holonomies which share the link $U(m,n;\hat{1})$ and parallel transport their electric field to the site (m,n). As the link $U(m,n;\hat{1})$ is shared by $\mathcal{W}(m,n)$ and $\mathcal{W}(m,n-1)$ and their electric fields $\mathcal{E}_-(m,n)$ and $\mathcal{E}_+(m,n-1)$ reside at sites (m,n) and $(m,n-1)$ respectively. Therefore $\mathcal{E}_+(m,n-1)$ has to be parallel transported by the holonomy $\mathcal{U}(m,n-1;\hat{2})$ leading to the relation (5.18b). These inverse relations are graphically illustrated in Figure 5.7-a,b,c respectively.

5.1.2 $(N\times N)$ plaquette lattice

We now generalize the dual relations obtained in the previous section to $N\times N$ lattice. There are N^2 horizontal links at $(m,n>0)$ as shown in Figure 5.1. Using (5.1) we canonically transform them into plaquettes in the clockwise direction as shown in Figure 5.5-b. This canonical glueing starts from the top left column and goes from the top to the bottom and then repeated iteratively in the adjacent right columns. As each plaquette formation requires 3 canonical transformations (see section B), we need a total of $3N^2$ iterative canonical transformations to cover the entire lattice. At the end we have (a) N^2 plaquettes pairs $(\mathcal{E}(\vec{n}),\,\mathcal{W}(\vec{n}))$ (b) all $N(N+1)$ vertical strings pairs $(\mathcal{E}(\vec{n};\hat{2}),\,\mathcal{W}(\vec{n};\hat{2}))$ and (c) N horizontal strings at $(\mathcal{E}(m,0;\hat{1}),\,\mathcal{W}(m,0;\hat{1}))$. These new configurations and their left, right electric fields are shown in Figure 5.5-b.

Plaquette, Strings & Duality

The N^2 plaquettes fluxes are:

$$\mathcal{W}(\vec{n}) = U(\vec{n};\hat{2})U(\vec{n}+\hat{2};\hat{1})U^\dagger(\vec{n}+\hat{1};\hat{2})U^\dagger(\vec{n};\hat{2}), \tag{5.20}$$

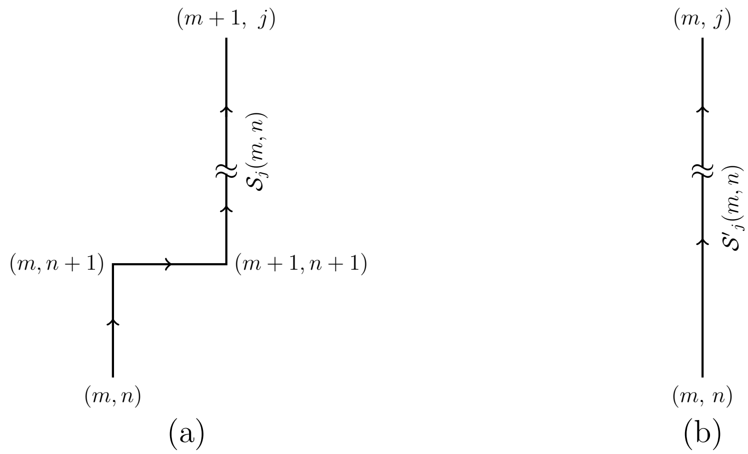


Figure 5.6: Two types of nonlocal strings required for canonical transformation: (a) $\mathcal{S}_j(m,n)$ defining the electric fields of $\mathcal{W}(m,n)$ in (5.21) and (5.24b), (b) $\mathcal{S}'_j(m,n)$ defining the electric fields of the strings $\mathcal{U}(m,n;\hat{i}\,)$ in (5.24b).

Their conjugate left electric fields are

$$\mathcal{E}_+(m,n) = -\sum_{j=n+1}^{N} \mathcal{S}_j(m,n)E_-(m+1,j;\hat{1})\ \mathcal{S}_j^{-1}(m,n). \tag{5.21}$$

In (5.21) $\vec{n} \equiv (m,n)$ and $m,n = 0,1,\cdots,N-1$. Canonical relations (5.21) is a straightforward generalisation of relation (5.6) replacing 2 by N which amounts to include all the horizontal Kogut Susskind electric fields $E_-(m+1,j>n;\hat{1})$ up to the top of the lattice. In (5.21) we have defined the parallel transport matrix operator $\mathcal{S}_j(m,n)$ shown in Figure 5.6-a

$$\mathcal{S}_j(m,n) \equiv U(m,n;\hat{2})U(m,n+1;\hat{1}) \prod_{k=n+1}^{j-1} U(m+1,k;\hat{2}). \tag{5.22}$$

In (5.22) $j \geq n+2$ and $\mathcal{S}_{n+1}(m,n) \equiv U(m,n;\hat{2})U(m,n+1;\hat{1})$. The nonlocal parallel transport operators $\mathcal{S}_j(m,n)$ encode the cumulative effects of all $3N^2$ canonical transformations over the entire lattice. As mentioned in the previous section, they are necessary for SU(N) gauge covariance of (5.22).

The asymmetry in the shape of the $\mathcal{S}_j(m,n)$ is because of the choice of iterative canonical transformations. In this work we started at the left top corner and proceeded toward the bottom in the first column and then moved to the adjacent right column. We know that $\mathcal{W}(m,n); n = (N-1),(N-2),\cdots 0$ are created sequentially by absorbing $U(m,n+1;\hat{1})$ at $(N-n)^{th}$ step starting from the top. Therefore its electric field must contain all $(N-n)$ Kogut Susskind electric fields on the horizontal links above it. They are located at different points and are parallel transported to (m, n) via path $\mathcal{S}$ to maintain gauge covariance of (5.21). The plaquette canonical commutation relations (5.8) and (5.9) discussed in the previous section on the simple 2×2 lattice remain valid. As before, under SU(N) gauge transformations (3.22) these plaquette conjugate pair transform as adjoint matter fields (5.10).

We now come to the string sector. The N horizontal and $N(N+1)$ vertical strings are related to old link variables as:

$$\mathcal{U}(m, n=0; \hat{1}) = U(m, 0; \hat{1}), \quad \mathcal{U}(\vec{n}; \hat{2}) = U(\vec{n}; \hat{2}). \tag{5.23}$$

Like in section 5.1.1, the conjugate electric fields are

$$\mathcal{E}_{+}(m, 0; \hat{1}) = E_{+}(m, 0; \hat{1}) - \sum_{j=1}^{N} \mathcal{S}_{j}(m, 0)\; E_{-}(m+1, j; \hat{1})\; \mathcal{S}_{j}^{-1}(m, 0), \tag{5.24a}$$

$$\begin{aligned}\mathcal{E}_{+}(m, n; \hat{2}) = \; & E_{+}(m, n; \hat{2}) - \sum_{j=n+1}^{N} \mathcal{S'}_{j}(m, n)\; E_{-}(m, j; \hat{1})\; \mathcal{S'}^{-1}_{j}(m, n) \\ & + \sum_{j=n+1}^{N} \mathcal{S}_{j}(m, n)\; E_{-}(m+1, j; \hat{1}) \mathcal{S}_{j}^{-1}(m, n).\end{aligned} \tag{5.24b}$$

As before, the vertical parallel transports are

$$\mathcal{S'}_{j}(m, n) = \prod_{k=n}^{j-1} U(m, k; \hat{2}) \tag{5.25}$$

and $\mathcal{S'}_{0}(m, 0) = 1$. The canonical transformations ensure that all $N(N+1)$ vertical strings pair $(\mathcal{E}^{\mathrm{a}}_{+}(\vec{n}; \hat{2}),\, \mathcal{U}_{\alpha\beta}(\vec{n}; \hat{2}))$ and N horizontal strings pair $(\mathcal{E}^{\mathrm{a}}_{+}(m, 0; \hat{1}),\, \mathcal{U}_{\alpha\beta}(m, 0; \hat{1}))$ satisfy the standard canonical commutations results. Under SU(N) gauge transformations (3.22) the string conjugate pairs transform as gauge fields (5.15).

Inverse relations

We can get Kogut-Susskind link operators from plaquette and string holonomies by solving equation (5.20) and (5.23);

$$U(\vec{n}; \hat{1}) = \mathsf{S}(\vec{n}; \hat{1}) \tag{5.26a}$$

$$U(\vec{n}; \hat{2}) = \mathcal{U}(\vec{n}; \hat{2}) \tag{5.26b}$$

Where $\mathsf{S}(\vec{n}; \hat{1})$ is the shortest path containing the plaquette holonomy $\mathcal{W}$ which connects the

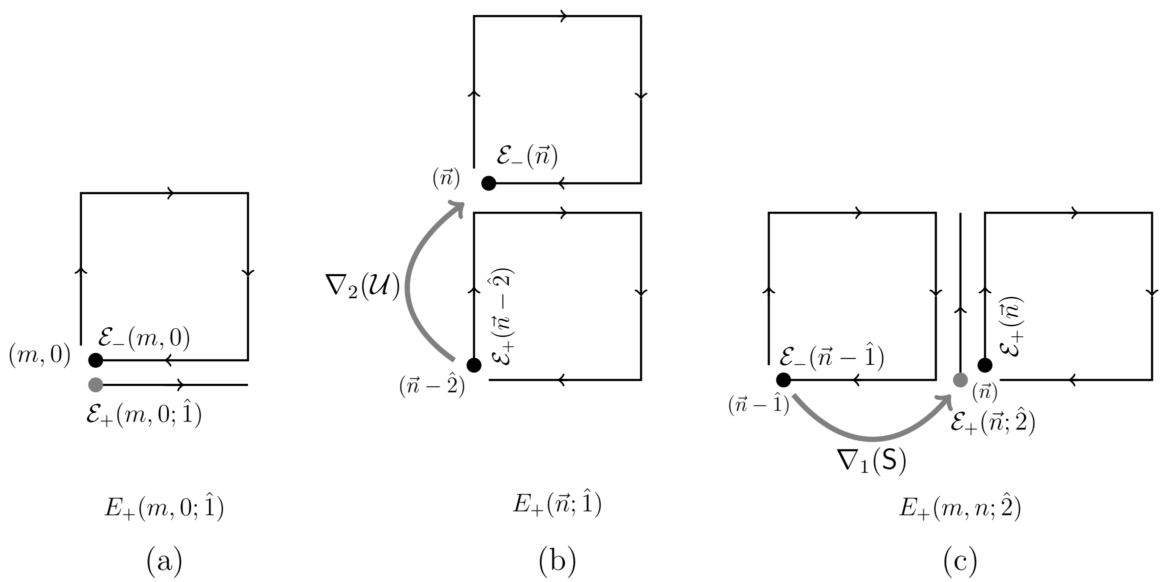


Figure 5.7: Inverse canonical relations: Kogut-Susskind electric fields in terms of the dual plaquette & String electric fields (a) Horizontal left electric field $E_+(m,0;\hat{1})$ in (5.18a), (b) Horizontal left electric fields $E_+(m,n;\hat{1})$, $n \neq 0$ in (5.18b) and (c) Vertical left electric field $E_+(m,n;\hat{2})$ in (5.18c). The plaquette electric fields $\mathcal{E}(m,n)$ are shown by $\bullet$ and string electric fields $\mathcal{E}(m,n;\hat{i})$ are shown by $\bullet$. The round arrows show that required parallel transports.

sites $\vec{n}$ and $\vec{n}+\hat{1}$ (see Figure 5.8-(a)):

$$\mathsf{S}(\vec{n};\hat{1}) = \mathcal{L}(\vec{n})\, \mathcal{U}(m,0;\hat{1})\, \mathcal{R}(\vec{n}+\hat{1}) \tag{5.27}$$

In (5.27), we have defined left and right parallel transports

$$\mathcal{L}(\vec{n}) \equiv \left[\prod_{j=1}^{n} \mathcal{U}^\dagger(m,n-j;\hat{2})\, \mathcal{W}(m,n-j)\right] \tag{5.28a}$$

$$\mathcal{R}(\vec{n}+\hat{1}) \equiv \left[\prod_{k=0}^{n-1} \mathcal{U}(m+1,k;\hat{2})\right] \tag{5.28b}$$

It is easy to invert the canonical relations (5.21), (5.24a) and (5.24b) to get the Kogut-Susskind electric fields in terms of the plaquette and string fields:

$$E_+(m,0;\hat{1}) = \mathcal{E}_-(m,0) + \mathcal{E}_+(m,0;\hat{1}) \tag{5.29a}$$

$$E_+(\vec{n};\hat{1}) = \nabla_2(\mathcal{U})\ \mathcal{E}(n) \tag{5.29b}$$

$$E_+(\vec{n};\hat{2}) = -\nabla_1(\mathcal{S})\ \mathcal{E}(\vec{n}) + \mathcal{E}_+(\vec{n};\hat{2}) \tag{5.29c}$$

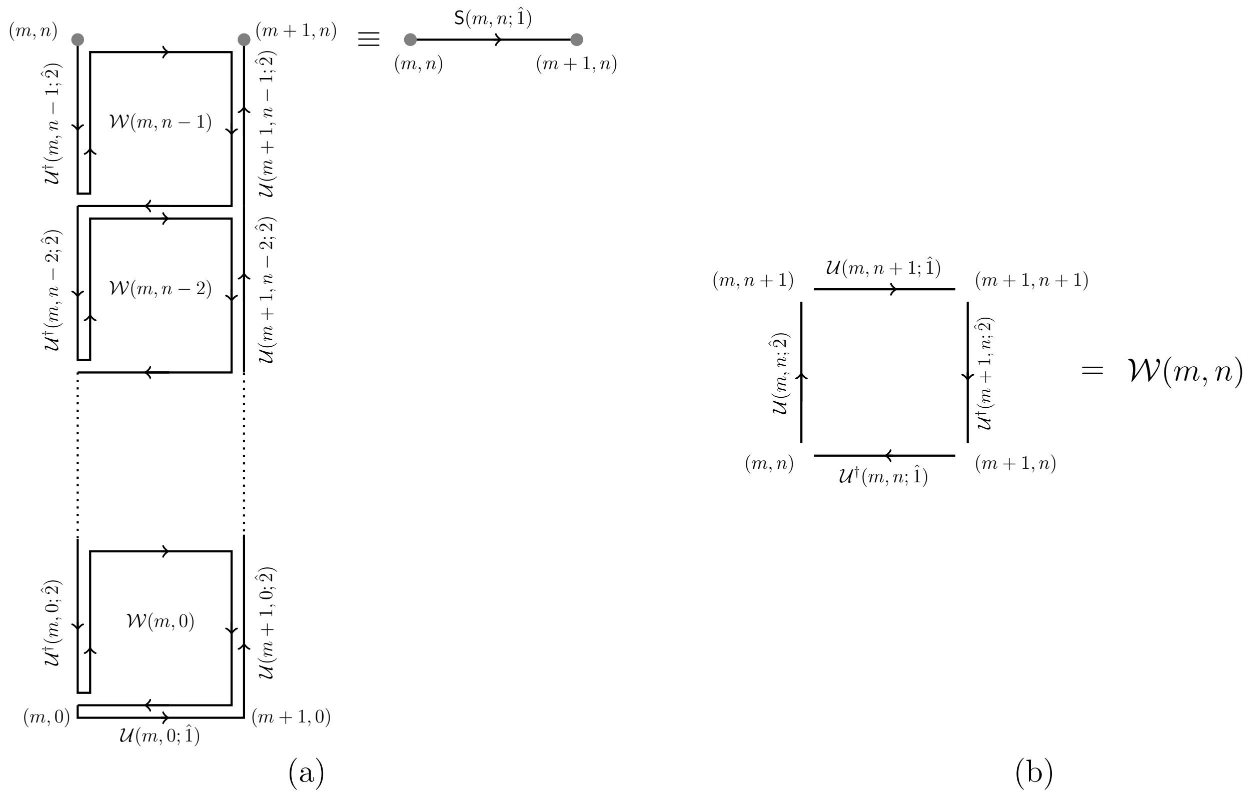


Figure 5.8: (a) Nonlocal interactions in the horizontal direction between $\mathcal{E}_-(\vec{n}-\hat{1})$ and $\mathcal{E}_+(\vec{n})$ in (5.31). The parallel transport denoted by $\mathsf{S}(m,n;\hat{1})$ depends nonlocally on $\mathcal{U}$ and $\mathcal{W}$ leading to nonlocal interactions in (5.31). (b) Introduction of new gauge fields $\mathcal{U}(m,n;\hat{1})$ through the local plaquette constraints (5.33) converts them into minimal couplings.

In the inverse duality relations (5.29a), (5.29b), (5.29c) we have defined the difference operators with local $\mathcal{U}(\vec{n}-\hat{2};\hat{2})$ and nonlocal $\mathsf{S}^\dagger\left(\vec{n}-\hat{1};\hat{1}\right)$ parallel transports as

$$\nabla_2(\mathcal{U})\mathcal{E}(\vec{n}) \equiv \mathcal{E}_-(\vec{n}) + \mathcal{U}^\dagger(\vec{n}-\hat{2};\hat{2})\mathcal{E}_+(\vec{n}-\hat{2})\mathcal{U}(\vec{n}-\hat{2};\hat{2}) \tag{5.30a}$$

$$\nabla_1(\mathcal{S})\mathcal{E}(\vec{n}) \equiv -\mathcal{E}_+(\vec{n}) - \mathsf{S}^\dagger(\vec{n}-\hat{1};\hat{1})\mathcal{E}_-(\vec{n}-\hat{1})\mathsf{S}(\vec{n}-\hat{1};\hat{1}). \tag{5.30b}$$

Note that after duality the Kogut electric fields are not fundamental. They are instead expressed in terms of electric scalar potentials. In the dual theory, these electric scalar potentials describe the gauge theory interactions as opposed to the magnetic vector potentials which describe interactions in the original Hamiltonian (3.20). This is discussed in the next section.

5.2 SU(N) Dual Dynamics

The Kogut-Susskind Hamiltonian (3.20) can now be rewritten in terms of the dual plaquette and string operators as [3]

$$H=\sum_{\vec{n}}\Big[g^2\mathrm{Tr}\Big((\nabla_2(\mathcal{U})\mathcal{E}(\vec{n}))^2+\big(\mathcal{E}_+(\vec{n},\hat{2})-\nabla_1(\mathsf{S})\,\mathcal{E}(\vec{n})\big)^2\Big)+\frac{K}{g^2}\ \big(2N-\mathrm{Tr}\,(\mathcal{W}(\vec{n})+\mathcal{W}^\dagger(\vec{n}))\big)\Big] \tag{5.31}$$

This dual or loop description is invariant under SU(N) gauge transformations (5.10), (5.15). It is also simple to interpret as follows. The original interacting magnetic field term in (3.20) which dominates near the $g^2 \to 0$ continuum limit is now simply the non-interacting magnetic field term $\frac{1}{g^2}\ \mathrm{Tr}(\mathcal{W}+\mathcal{W}^\dagger) \sim \frac{1}{g^2}\ \vec{B}^2$ term. This is one of the expected outcomes of duality transformations. On the other hand, the original non-interacting electric field terms in (3.20) now describe the interactions in terms of the electric scalar potentials which transform like SU(N) adjoint matter field (5.10). Infact, it is clear that the dual interaction terms in (5.31) correspond to minimal coupling between adjoint electric scalar potentials and the SU(N) gauge fields. The Gauss law constraints associated with the SU(N) gauge invariance (5.10) and (5.15) are

$$\mathcal{G}^{\mathrm{a}}(\vec{n})=\mathcal{E}^{\mathrm{a}}_-(\vec{n})+\mathcal{E}^{\mathrm{a}}_+(\vec{n})+\mathcal{E}^{\mathrm{a}}_+(\vec{n};\hat{2})+\mathcal{E}^{\mathrm{a}}_-(\vec{n};\hat{2})=0. \tag{5.32}$$

The above constraints follow from the new configurations in Figure 5.3-b. It is easy to check that the new Gauss law constraints (5.32) reduce to the old Gauss law constraints (3.22) when the canonical relations are used and thus confirming (5.21) and (5.24a), (5.24b). The immediate problem we face is that the interactions in the dual description (5.31) are nonlocal and asymmetric (compare (5.29b) with (5.29c)) due to the presence of $\mathsf{S}(\vec{n};\hat{1})$ in (5.29c). The underlying reason for this nonlocality and asymmetry is simply the absence of the horizontal holonomies which have been transformed into $\mathcal{W}(m,n)$ by canonical transformations and shown in Figure 5.3-b. In the next section, we will remove these obstacles.

5.2.1 Plaquette constraints

Having obtained the dual magnetic field description, we resolve the above asymmetry and nonlocality problems by the reintroduction of horizontal link holonomies $\mathcal{U}(\vec{n};\hat{1})$ through the local plaquette constraints:

$$\mathcal{U}(\vec{n};\hat{2})\,\mathcal{U}(\vec{n}+\hat{2};\hat{1})\,\mathcal{U}^\dagger(\vec{n}+\hat{1};\hat{2})\,\mathcal{U}^\dagger(\vec{n};\hat{1})=\mathcal{W}(\vec{n}). \tag{5.33}$$

[3] From now onward we assume $N \to \infty$ limit and ignore the boundary effects.

Note that the constraints (5.33) imposed on the dual theory are consistent with the dual gauge transformations (5.10) and (5.15). They physically mean that the newly created gauge invariant Wilson loops with gauge fields $(\mathcal{U}(\vec{n};\hat{1}), \mathcal{U}(\vec{n};\hat{2}))$ do not lead to any additional physical degrees of freedom. The motivation for introducing (5.33) is that on the constrained surface

$$\mathcal{U}(\vec{n};\hat{1}) = \mathsf{S}(\vec{n};\hat{1}). \tag{5.34}$$

Now the nonlocal inverse relation (5.29c) takes the local form and we write

$$E_{+}(\vec{n};\hat{i}) = \delta_{i2}\mathcal{E}_{+}(\vec{n};\hat{i}) + \epsilon_{ij}\nabla_{j}(\mathcal{U})\mathcal{E}(\vec{n}). \tag{5.35}$$

In (5.35) $i, j = 1, 2$. The plaquette constraints (5.33) must commute with the Hamiltonian H in (5.31). It is clear that the magnetic part, $H_M \sim \mathrm{Tr}\,\mathcal{W}(\vec{n}))$, commutes with (5.33) as $\mathcal{W}_{\alpha\beta}(\vec{n})$ and $\mathcal{U}_{\alpha\beta}(\vec{n};\hat{i})$ are mutually independent and commuting dual degrees of freedom. It is easy to see that the constraints (5.33) will commute with the electric part H_E ($H_E \sim \vec{E}^2(\vec{n};\hat{1}) + \vec{E}^2(\vec{n};\hat{2})$) also if the electric fields $E^{\mathrm{a}}_{+}(\vec{n};\hat{1})$ and $E^{\mathrm{a}}_{+}(\vec{n};\hat{2})$ defined by (5.29b) and (5.29c) rotate both sides of (5.33) covariantly. We therefore introduce electric fields $\mathcal{E}_{+}(\vec{n};\hat{1})$ which are conjugate to auxiliary gauge fields $\mathcal{U}(\vec{n};\hat{1})$ and write

$$E_{+}(\vec{n};\hat{i}) = \mathcal{E}_{+}(\vec{n};\hat{i}) + \epsilon_{ij}\nabla_{j}(\mathcal{U})\,\mathcal{E}(\vec{n}). \tag{5.36}$$

In (5.36) the covariant derivatives are defined as

$$\nabla_2(\mathcal{U})\mathcal{E}(\vec{n}) \equiv \mathcal{E}_{-}(\vec{n}) + \mathcal{U}^{\dagger}(\vec{n}-\hat{2};\hat{2})\mathcal{E}_{+}(\vec{n}-\hat{2})\mathcal{U}(\vec{n}-\hat{2};\hat{2}) \tag{5.37a}$$

$$\nabla_1(\mathcal{U})\mathcal{E}(\vec{n}) \equiv -\mathcal{E}_{+}(\vec{n}) - \mathcal{U}^{\dagger}(\vec{n}-\hat{1};\hat{1})\mathcal{E}_{-}(\vec{n}-\hat{1})\mathcal{U}(\vec{n}-\hat{1};\hat{1}) \tag{5.37b}$$

As mentioned before the parallel transports in (5.37a) and (5.37b) are required for SU(N) gauge covariance.

At this stage it is interesting as well as illustrative to compare the equation (5.36) with the corresponding equation (2.69) in U(1) or Z(N) lattice gauge theories [29, 38, 48, 49, 67, 69–79]. In U(1) case the Gauss law constraints in $(2 + 1)$ dimension are:

$$\vec{\nabla}\cdot\vec{E}(\vec{n}) \equiv \sum_{i=1}^{2}\Big(\nabla_i E(\vec{n};\hat{i})\Big) = 0, \tag{5.38}$$

where ∇_i is the simple difference operator in $i = 1, 2$ directions. The solutions are in terms of the Abelian electric scalar potentials:

$$E(\vec{n};\hat{i}) = \epsilon_{ij}\nabla_j\mathcal{E}(\vec{n}). \tag{5.39}$$

The SU(N) electric scalar potential defining equations (5.36) are an obvious generalization of

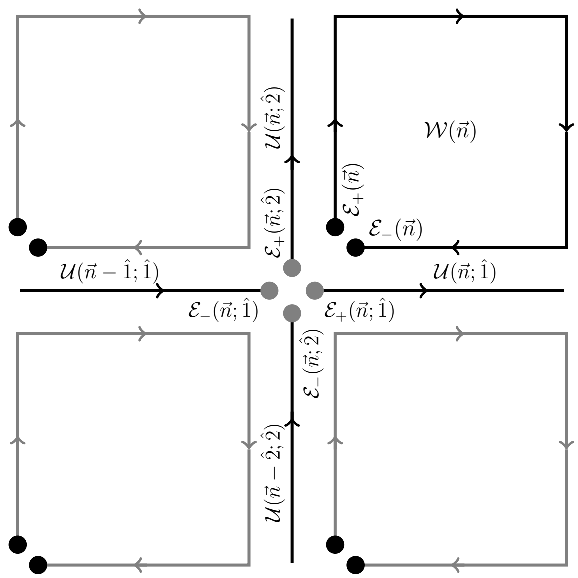


Figure 5.9: New Gauss Law $\mathcal{E}^{\rm a}_-(\vec{n}) + \mathcal{E}^{\rm a}_+(\vec{n}) + \sum_{i=1}^{2}\left(\mathcal{E}^{\rm a}_+(\vec{n};\hat{i}) + \mathcal{E}^{\rm a}_-(\vec{n};\hat{i})\right) = 0$. There is six electric fields at each site $\vec{n}$. The two plaquette electric fields or electric scalar potentials are shown by dark bullets • and the four string electric fields are shown by grey bullets •.

the corresponding Abelian equations (5.39) (or (2.69)) where the ordinary difference operators get replaced by the SU(N) covariant difference operators. In the present, duality through canonical transformations context, the parallel transports in (3.14) and (5.30a), (5.30b) in the simple U(1) case is Abelian phase factors and cancel out. Thus there are no link gauge fields and we recover (5.39) without any nonlocality problem. The SU(N) Gauss law constraints

$$\mathcal{G}^{\rm a}(\vec{n}) = \mathcal{E}^{\rm a}_-(\vec{n}) + \mathcal{E}^{\rm a}_+(\vec{n}) + \sum_{i=1,2}\left(\mathcal{E}^{\rm a}_-(\vec{n};\hat{i}) + \mathcal{E}^{\rm a}_+(\vec{n};\hat{i})\right) = 0. \tag{5.40}$$

are symmetric now, see Figure 5.9. Under SU(N) gauge transformations all electric fields appearing in (5.40) transform like adjoint matter fields.

The new Hamiltonian which commutes with the constraints (5.33) written in terms of the dual operators is

$$H = \sum_{\vec{n}}\left[g^2 \,\mathrm{Tr}\sum_{i=1}^{2}\left(\mathcal{E}_+(\vec{n},\hat{i}) + \epsilon_{ij}\nabla_j(\mathcal{U})\,\mathcal{E}(\vec{n})\right)^2 + \frac{K}{g^2}\,\left(2\mathrm{N} - \mathrm{Tr}\left(\mathcal{W}(\vec{n}) + \mathcal{W}^\dagger(\vec{n})\right)\right)\right] \tag{5.41}$$

The dual Hamiltonian (5.41) can also be interpreted as the loop Hamiltonian. It's physical interpretation is very simple. The second interacting term in (3.20) dualizes to the non-interacting magnetic field term in (5.41). It creates & annihilates the single plaquette loops. This is most transparent in the prepotential operator language [61, 62, 78, 79, 137]. The first original non-interacting electric field term dualizes to the loop-loop interaction term. These loop interactions are through minimal couplings to the gauge fields. This duality between interacting and non-interacting terms leads to the inversion of the coupling constant: $g^2 \rightarrow \frac{1}{g^2}$. Note that the physical degrees of freedom are associated only with the SU(N) magnetic fields and their conjugate electric potentials $(\mathcal{E}(\vec{n}),\ \mathcal{W}(\vec{n}))$. The auxiliary string sector $(\mathcal{E}(\vec{n};\hat{i}),\ \mathcal{U}(\vec{n};\hat{i}))$ makes the dual description local as well as simple. Infact, the SU(N) duality transformations leading to dual SU(N) spin model without any gauge or string degrees of freedom have been studied in the past [78, 79]. They lead to non local dynamics. In the present framework, with all interactions local and proportional to g^2, the dual Hamiltonian see (5.41) can be used to set up a weak coupling perturbation theory near the continuum $g^2 \rightarrow 0$ limit. The matter fields can be coupled to the SU(N) gauge fields $\mathcal{U}(\vec{n};\hat{i})$ through minimal coupling so that the SU(N) gauge invariance (5.10) and (5.15) remains intact.

CHAPTER 6

SU(N) DISORDER OPERATOR

Disorder operators were first discussed by Kadanoff in the context of the two-dimensional Ising model [39]. We have briefly discussed them in Chapter 2 and showed that they have non-zero expectation values in the disordered phase. As the QCD vacuum is supposed to be magnetically disordered, it is important to construct and analyse these operators in complete detail in non-Abelian gauge theories.

In this chapter, we construct the most general disorder operator for SU(N) lattice gauge theory in (2+1) dimension by exploiting exact non-Abelian duality transformations discussed in Chapter 4. The corresponding most general order-disorder algebra is also derived and discussed. The 't Hooft disorder operator is obtained as a special limit of this most general disorder operator. As expected, in this limit we also recover Wilson-'t Hooft order-disorder algebra. Finally, we briefly discuss the role of these operators in the phase transition of the theory. It is known that the disorder operators are the natural and local operators in terms of the dual operators. The Kramers–Wannier duality in $(1 + 1)$ dimensional Ising model (see Section 2.1) leading naturally to the disorder operator is the simplest example which illustrates this particular aspect or advantage of duality transformations [37]. In this model, the disorder operators are simply the dual-spin operators which describe the dual interactions with inverse coupling. They also create Z_2 kinks which are responsible for disordering the ground state leading to the loss of magnetization above the Curie temperature.

In gauge theory, the disorder operators acquire additional meaning and significance as the underlying duality transformations also interchange the roles of electric and magnetic degrees of freedom. Again, the Wegner duality in the simple $Z_{\rm N}$ Ising gauge theories in $(2 + 1)$ and $(3 + 1)$ dimensions clearly illustrates this additional this additional rich feature [39, 94]. More explicitly, in $(2 + 1)$ dimension Z_2 lattice gauge theory the disorder operators are the dual spin or dual Z_2 electric potential operators [93, 94, 98] which describe the interactions in the dual formulation with inverse coupling. Being conjugate to the Z_2 magnetic fields, they also create Z_2 magnetic vortices. These vortices, in turn, magnetically disorder the ground states in the confining phase [93, 98] and are thus responsible for the confinement-deconfinement phase transition.

In general, the order (disorder) operators are related to the potentials (dual potentials) which are conjugate to electric (magnetic) fields respectively. They can therefore be interpreted as the "translation operators" for the electric and magnetic fluxes respectively. Moreover, the order-disorder algebra is simply the canonical commutation relations between the dual conjugate operators, i.e, between the magnetic flux and the electric potential operators (see the relations (6.1), (6.2) and (6.3)). In SU(3) lattice gauge theory or QCD the color confinement can be viewed as a consequence of magnetically disordered ground state leading to area law for the Wilson loops. Like the various cases discussed above, the magnetic disorder in QCD is produced by the magnetic vortices, which in tun are created by the SU(3) disorder operators leading to disordered ground state. A systematic study of these disorder operators in SU(2), SU(3) and then SU(N) lattice gauge theories, using exact duality transformations, is the subject of this work.

In 1978 't Hooft emphasised the role of disorder operators in the context of quark confinement in SU(N) gauge theory [95]. The 't Hooft disorder operator creates topological charges or magnetic fluxes which belong to the centre Z_N of the gauge group SU(N). They have been extensively studied in the past in the weak coupling limit [94, 96, 97, 154, 155]. It is well known that the SU(N) 't Hooft loop operators are dual to the SU(N) Wilson loop order operators in a limited sense [97, 156] as they create only the centre or Z_N magnetic fluxes. More explicitly, while SU(N) Wilson loop operators are characterized by the $(\mathrm{N}-1)$ Casimir operators or equivalently a set of $(\mathrm{N}-1)$ integers, (see [61, 139]), the SU(N) 't Hooft disorder operator is labelled by a single Z_N quantum number taking $0, 1, \cdots, (\mathrm{N}-1)$ values.

In this chapter, we construct the most general disorder operator for SU(N) lattice gauge theory in $(2+1)$ dimensions by exploiting the exact duality transformations [78, 79, 91]. These disorder operators $\Sigma_{[\vec{\theta}]}(p)$ are defined on plaquettes p as:

$$\Sigma^{\pm}_{[\vec{\theta}]}(p) = \exp\ i\big(\vec{\theta}(p)\cdot\vec{\mathcal{E}}_{\pm}(p)\big). \tag{6.1}$$

In (6.1), $\vec{\mathcal{E}}_{\pm}(p)$ are the SU(N) "electric scalar potentials" on the plaquette p. They are related to the SU(N) electric fields through the exact duality transformations (6.4). The SU(N) disorder operator $\Sigma^{\pm}_{[\vec{\theta}]}(p)$ in (6.1) is characterized by a set of $(\mathrm{N}-1)$ angles which are denoted by $[\vec{\theta}] \equiv (\theta_1(p), \theta_2(p), \cdots, \theta_{\mathrm{N}-1}(p))$ on each plaquette. In this work, like the Kramers-Wannier spin and Wegner gauge dualities discussed earlier, we show that the exact SU(N) duality transformations naturally lead to $\Sigma^{\pm}_{[\vec{\theta}]}(p)$ in (6.1). We further show that they are the creation & annihilation operators for the SU(N) magnetic votices on the spatial plaquettes.

The Wilson loop order operators $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$, on the other hand, are defines as a path-ordered product of the link holonomies along a directed loop $\mathcal{C}$:

$$\mathcal{W}^{[\vec{j}]}(\mathcal{C}) = \prod_{l \in \mathcal{C}} U^{[\vec{j}]}(l). \tag{6.2}$$

In (6.2), $U^{[\vec{j}]}(l)$ are the SU(N) link holonomies or the "magnetic vector potentials" in a general $[\vec{j}]$ representation of SU(N). Note that the SU(N) order operator $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$ is characterized by a set of (N − 1) integers on loop $\mathcal{C}$ and $[\vec{j}] \equiv (j_1, j_2, \cdots, j_{\mathrm{N}-1})$. The representation index $[\vec{j}]$ denotes the (N-1) eigenvalues $(j_1, j_2, \cdots, j_{\mathrm{N}-1})$ of the (N-1) SU(N) Casimir operators. These Casimir operators (constructed purely out of the electric field operators) acting on the SU(N) electric basis measure the net electric fluxes on the loop states created by the loop operator Tr $(\mathcal{W}^{[\vec{j}]}(\mathcal{C}))$. In this work we also obtain the SU(N) order-disorder operators algebra:

$$\Sigma_{[\vec{\theta}]}(p) \ \left(\mathcal{W}^{[\vec{j}]}(\mathcal{C})\right)_{\alpha\beta} \ \Sigma^{-1}_{[\vec{\theta}]}(p) = \begin{cases} \left(D^{[\vec{j}]}(\vec{\theta}) \ \mathcal{W}^{[\vec{j}]}(\mathcal{C})\right)_{\alpha\beta}, & \text{if } p \text{ inside } \mathcal{C} \\ \left(\mathcal{W}^{[\vec{j}]}(\mathcal{C})\right)_{\alpha\beta}, & \text{otherwise.} \end{cases} \tag{6.3}$$

In (6.3), $D^{[\vec{j}]}(\vec{\theta})$ denotes the SU(N) Wigner rotation matrix in the $[\vec{j}]$ representation. If the angles $[\vec{\theta}]$ correspond to the centre element $z \in Z_{\mathrm{N}}$ with $z^{\mathrm{N}} = 1$, then using [1] $D^{[\vec{j}]}(z) = (z)^{\eta[\vec{j}]}$, where $\eta[\vec{j}]\ (= 0, 1, 2 \cdots, (\mathrm{N}-1))$ is the N-ality of the representation $[\vec{j}]$, we recover the standard 't Hooft-Wilson order-disorder algebra discussed in [95, 116].

The plan of the chapter is as follows. We begin with summarising the relevant SU(N) duality transformations obtained in chapter 4. The SU(N) magnetic vortex creation and annihilation or equivalently SU(N) disorder operators are discussed in section 6.1. In order to simplify the presentation, the SU(2), SU(3) and SU(N) disorder operators are discussed one by one in the increasing order of difficulty in sections 6.1, 6.1 and 6.1 respectively. In the simplest SU(2) case, we construct the magnetic basis in Section 6.1-A using the SU(2) prepotential approach [61]. In section 6.1-B we show that the SU(2) disorder operators act as SU(2) magnetic vortex creation-annihilation operators on the magnetic basis. The SU(2) order-disorder algebra is discussed in 6.1-C. We then consider the SU(3) case in detail in 6.1. As expected, there are many new SU(3) features which are absent in the simple SU(2) case. In particular, we emphasize the importance of the SU(3) prepotential operators representation of the dual electric scalar potentials for constructing the SU(3) magnetic fields. In section 6.1 we directly generalize these SU(3) results to the SU(N) case. In section 6.2, we rewrite the SU(N) disorder operator in the

[1]This result follows from the SU(N) Young tableau in the $[\vec{j}]$ representation with total L fundamental boxes. If each of them is rotated by the center element z then we get $D^{[\vec{j}]}(z) = (z)^L\ \mathcal{I} = (z)^{\eta[\vec{j}]}\ \mathcal{I}$ as $z^{\mathrm{N}} = 1$. Here $\mathcal{I}$ is the identity matrix in the $[\vec{j}]$ representation of SU(N).

original Kogut-Susskind formulation. We show that they now become non-local operators and are attached with the invisible SU(N) Dirac strings. As expected, these unphysical strings can be moved around by SU(N) gauge transformations without changing their end points which specify the locations of the SU(N) gauge invariant magnetic vortices and anti-vortices. In Sec. 6.3 we compute the path integral expression for the SU(N) vortex-free energy. This path integral representation should be useful for Monte Carlo simulations and to understand the role of these magnetic vortices and their condensation, if any, in the color confinement problem. It is expected that they will condense and disorder the vacuum state for any non-zero coupling constant.

The prepotential operators create and annihilate the SU(N) electric as well as the magnetic fluxes [61]. Therefore they provide a common platform to construct both the electric and magnetic bases in the physical loop Hilbert space of SU(N) lattice gauge theory. In these two dual bases we show that the order and disorder operators have natural action of translating the electric and magnetic fluxes respectively. These SU(N) electric and magnetic bases and the action of the order and the disorder operators on them are discussed in detail in Appendix C and Appendix D respectively. Appendix E shows that the SU(N) Dirac strings are unphysical.

In chapter4, we had reformulated SU(N) lattice gauge theory in terms of SU(N) spin degrees of freedom through a series of canonical transformations over the entire lattice in $(2+1)$ dimension. The dual model is written in terms of the mutually independent plaquette loops (see Figure 6.1-b) or scalar magnetic flux operators $\mathcal{W}(p)$ and their conjugate electric scalar potential $\vec{\mathcal{E}}(p)$ operators satisfying (6.7). We briefly recapitulate the dual SU(N) physical and unphysical operators in the following two subsections respectively.

1. Magnetic flux operators $(\mathcal{W}_{\alpha\beta}(p),\ \mathcal{E}^{\mathrm{a}}_{+}(p))$

They are the physical magnetic operators which solve the SU(N) Gauss law constraints and define the physical Hilbert space $\mathcal{H}^{\mathrm{phys}}$. They represent the scalar SU(N) magnetic fluxes $(\mathcal{W}(p))$ on plaquette p and their conjugate electric scalar potentials $\mathcal{E}_{\pm}(p)$ [2]. The SU(N) duality relations are

$$\begin{aligned}\mathcal{W}(m,n) &= \mathsf{T}(m-1,n-1)\ U_p(m,n)\ \mathsf{T}^{\dagger}(m-1,n-1)\\ \mathcal{E}_{+}(m,n) &= \sum_{n'=n}^{\infty} S^{\dagger}(m,n;n')E_{-}(m,n';\hat{1})S(m,n;n')\end{aligned} \tag{6.4}$$

The parallel transport operators $\mathsf{T}(m-1,n-1)$ and $S(m,n;n')$ are defined as (see Figure 6.1-c

[2] The convention chosen for loop (string) electric fields is that $\mathcal{E}^{\mathrm{a}}_{-}(\vec{n})$ $(\mathbb{E}^{\mathrm{a}}_{-}(\vec{n}))$ and $\mathcal{E}^{\mathrm{a}}_{+}(\vec{n})$ $(\mathbb{E}^{\mathrm{a}}_{+}(\vec{n}))$ are located at the initial, end points of the flux loop (string) respectively.

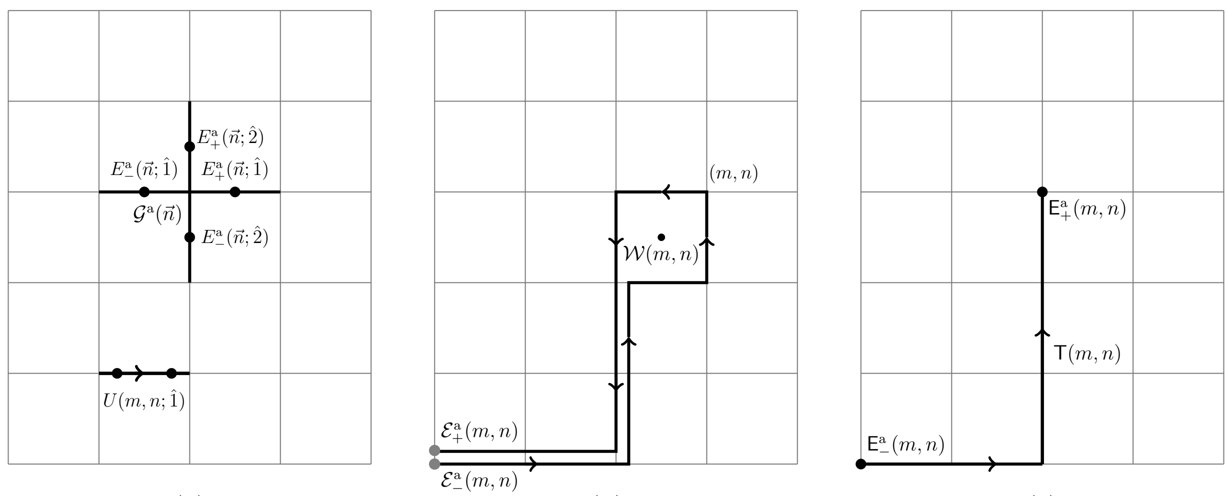


Figure 6.1: (a) Kogut Susskind link formulations. Link operator $U(\vec{n};\hat{i})$ and its left (right) $E^{\rm a}_+(\vec{n};\hat{i})$ $(E^{\rm a}_-(\vec{n}+\hat{i};\hat{i}))$ electric field. Gauss operator site $\vec{n}$, $\mathcal{G}(\vec{n}) = \sum_{i=1}^{2}[E^{\rm a}_+(\vec{n};\hat{i}) + E^{\rm a}_-(\vec{n};\hat{i})]$ is also shown, (b) Dual physical plaquette holonomy $\mathcal{W}(\vec{n})$ and their left (right) $\mathcal{E}_+(\vec{n})$ $(\mathcal{E}_-(\vec{n}))$ electric field, (c) Unphysical string holonomy $\mathsf{T}(m,n)$ and it's left $\mathsf{E}_+(\vec{n})(=\mathcal{G}(\vec{n}))$ and right $(\mathsf{E}_-(\vec{n}))$ electric field respectively. These string holonomies decouple on physical Hilbert space.

and Figure 4.3) are

$$\mathsf{T}(m,n) = \prod_{m'=0}^{m} U(m',0\,;\,\hat{1}) \prod_{n'=0}^{n} U(m,n';\,\hat{2}), \tag{6.5a}$$

$$S(m,n;n') \equiv \mathsf{T}(m{-}1,n)\, U(m{-}1,n;\hat{1}) \prod_{h=n}^{n'} U(m,h;\hat{2}). \tag{6.5b}$$

Like in the Kogut-Susskind approach, the right electric potentials are defined by

$$\mathcal{E}_-(p) = -\mathcal{W}^\dagger(p)\,\mathcal{E}_+(p)\,\mathcal{W}(p), \tag{6.6}$$

Note that $\mathcal{E}^{\rm a}_-(p)$ are attached to the initial end of plaquette flux line $\mathcal{W}(p)$ as shown in Figure 6.1-b. The dual operator commutation relations are

$$\left[\mathcal{E}^{\rm a}_+(p), \mathcal{W}_{\alpha\beta}(p)\right] = \left(T^{\rm a}\mathcal{W}(p)\right)_{\alpha\beta}, \qquad \left[\mathcal{E}^{\rm a}_-(p), \mathcal{W}_{\alpha\beta}(p)\right] = -\left(\mathcal{W}(p)\,T^{\rm a}\right)_{\alpha\beta}. \tag{6.7}$$

The above commutation relations imply that $\mathcal{E}^{\rm a}_+(p)$ $(\mathcal{E}^{\rm a}_-(p))$ rotate $\mathcal{W}_{\alpha\beta}(p)$ from left (right) and therefore are the left (right) electric scalar potentials. They are mutually independent and satisfy SU(N) algebra:

$$\left[\mathcal{E}^{\rm a}_{+}(p),\ \mathcal{E}^{\rm b}_{-}(p)\right] =\ 0, \qquad\qquad \left[\mathcal{E}^{\rm a}_{\pm}(p),\ \mathcal{E}^{\rm b}_{\pm}(p)\right] =\ i\, f^{\rm abc}\mathcal{E}^{\rm c}_{\pm}(p). \tag{6.8}$$

Also, the relation (6.6) implies that their magnitudes are equal:

$$\vec{\mathcal{E}}^{\,2}_{+}(p) = \vec{\mathcal{E}}^{\,2}_{-}(p) \equiv \vec{\mathcal{E}}^{\,2}(p), \tag{6.9}$$

In the first two equations above we have defined $\vec{\mathcal{E}}^{\,2}_{\pm}(p) \equiv \sum_{\rm a=1}^{\rm N^2-1} \mathcal{E}^{\rm a}_{\pm}(p)\mathcal{E}^{\rm a}_{\pm}(p)$. The dual spin or magnetic flux operators transform as SU(N) adjoint matter field at the origin

$$\mathcal{W}(m,n) \rightarrow \Lambda(0,0)\,\mathcal{W}(m,n)\Lambda^{\dagger}(0,0), \qquad \mathcal{E}_{\pm}(m,n) \rightarrow \quad \Lambda(0,0)\,\mathcal{E}_{\pm}(m,n)\Lambda^{\dagger}(0,0) \tag{6.10}$$

2. String operators $(\mathsf{E}^{\rm a}_{-}(\vec{n}),\ \mathsf{T}(\vec{n}))$

They are unphysical operators and represent SU(N) gauge degrees of freedom at every lattice site away from the origin. They are shown in Figure 6.1-c.

$$\begin{aligned} \mathsf{T}(m,n) &= \prod_{m'=0}^{m} U(m',0;\ \hat{1}) \prod_{n'=0}^{n} U(m,n';\ \hat{2}), \\ \mathsf{E}^{\rm a}_{+}(m,n) &=\ \mathcal{G}^{\rm a}(m,n) \simeq 0. \end{aligned} \tag{6.11}$$

Thus all string operators $\mathsf{T}(m,n)$ become cyclic as their conjugate electric fields $\mathsf{E}^{a}_{+}(m,n)$ turns out to be the Gauss law operator $\mathcal{G}^{a}(m,n)$ [78, 79]. Therefore they vanish on the physical Hilbert space $\mathcal{H}^{p}$ where the SU(N) Gauss laws are satisfied. The string operators, being unphysical, are irrelevant for defining disorder operators and therefore will not be considered henceforth.

6.1 Disorder operators

As mentioned earlier, the order and disorder operators in SU(N) lattice theory are simply the shift or the creation-annihilation operators for the gauge invariant electric and magnetic fluxes respectively. Note that the Wilson loop operators $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$, constructed in terms of the magnetic vector potentials $U(l)$ in (6.2), shift their conjugate electric fluxes along the loop $\mathcal{C}$. In this section, we construct the gauge invariant disorder operators which are dual to the Wilson loop operators $\mathcal{W}^{[\vec{j}]}(\mathcal{C})$ and shift the magnetic fluxes instead. For the sake of simplicity, we first consider SU(2) case and then generalize it to SU(3) and finally to the SU(N) gauge group.

SU(2) Disorder Operator

The magnetic plaquette flux operator,

$$\mathcal{W}^{[j=\frac{1}{2}]}(p) \equiv \exp \frac{i}{2} \left(\hat{n}(p) \cdot \vec{\sigma} \;\; \omega(p)\right). \tag{6.12}$$

can also be rewritten in the angle-axis representation as:

$$\mathcal{W}^{[j=\frac{1}{2}]}(p) \equiv \cos\left(\frac{\omega(p)}{2}\right)\sigma_0 + i\left(\hat{n}(p)\cdot\vec{\sigma}\right)\sin\left(\frac{\omega(p)}{2}\right); \quad \hat{n}(p)\cdot\hat{n}(p) = 1, \quad \forall\ (p). \tag{6.13}$$

In (6.13) $\sigma_0, \vec{\sigma}(\equiv \sigma_1, \sigma_2, \sigma_3)$ are the unit, 3 Pauli matrices respectively. Under global gauge transformation $\Lambda \equiv \Lambda(0,0)$ in (6.10), $(\omega, \hat{n})$ transform as:

$$\omega(p) \to \omega(p), \qquad \hat{n}(p) \equiv \sum_{\mathrm{a}=1}^{3} \hat{n}^{\mathrm{a}}(p)\sigma^{\mathrm{a}} \;\; \to \Lambda(0,0)\ \hat{n}(p)\ \Lambda^{\dagger}(0,0). \tag{6.14}$$

Thus $\omega(p)$ are gauge invariant angle and $\hat{n}(p)$ are the unit vector operators. We now define two unitary operators:

$$\begin{aligned}\Sigma^{+}_{\theta}(p) &\equiv \exp\ i\Big(\ \hat{n}(p)\cdot\mathcal{E}_{+}(p)\ \theta\ \Big)\\ \Sigma^{-}_{\theta}(p) &\equiv \exp\ i\Big(\ \hat{n}(p)\cdot\mathcal{E}_{-}(p)\ \theta\ \Big),\end{aligned} \tag{6.15}$$

which are located on a plaquette p. They both are gauge invariant because $\mathcal{E}^{\mathrm{a}}_{\pm}(p)$ and $\hat{n}(p)$ gauge transform like vectors as shown in (6.10) and (6.14). In other words, $\left[\mathcal{G}^{\mathrm{a}}, \Sigma^{\pm}_{\theta}(p)\right] = 0$, where $\mathcal{G}^{\mathrm{a}}$ is defined in (3.22). As the left and right electric scalar potentials are related through (6.6), $\Sigma^{\pm}_{\theta}(p)$ are not mutually independent and satisfy [3]:

$$\Sigma^{+}_{\theta}(p)\ \Sigma^{-}_{\theta}(p) = \Sigma^{-}_{\theta}(p)\ \Sigma^{+}_{\theta}(p) = \mathcal{I}. \tag{6.16}$$

In (6.16), $\mathcal{I}$ denotes the unit operator in the physical Hilbert space $\mathcal{H}^p$. The identities (6.16) can be easily obtained by using $\mathcal{E}_{-}(p) = -R^{\mathrm{ab}}(\hat{n},\omega)\mathcal{E}_{+}(p)$ and $R^{\mathrm{ab}}(\hat{n},\omega)\hat{n}^{\mathrm{b}} = \hat{n}^{\mathrm{a}}$ where $R^{\mathrm{ab}}(\hat{n},\omega) = \frac{1}{2}\mathrm{Tr}(\sigma^{\mathrm{a}}\mathcal{W}\sigma^{\mathrm{b}}\mathcal{W}^{\dagger})$.

---

[3] We have used the relation $n^{\mathrm{a}}(p)\mathcal{E}^{\mathrm{a}}_{-}(p) = -n^{\mathrm{a}}(p)R^{\mathrm{ab}}(\mathcal{W}^{\dagger}(p))\mathcal{E}^{\mathrm{b}}_{+}(p)$ and

$$\begin{aligned} n^{\mathrm{a}}(p)R^{\mathrm{ab}}(\mathcal{W}^{\dagger}(p)) = \mathrm{Tr}(\sigma^{\mathrm{a}}\mathcal{W}(p))\ \frac{1}{2}\mathrm{Tr}(\sigma^{\mathrm{a}}\mathcal{W}^{\dagger}(p)\sigma^{\mathrm{b}}\mathcal{W}(p)) &= \frac{1}{2}(\sigma^{\mathrm{a}}_{\alpha\beta}\sigma^{\mathrm{a}}_{\gamma\delta})\sigma^{\mathrm{b}}_{\eta\rho}\mathcal{W}_{\beta\alpha}(p)\mathcal{W}^{\dagger}_{\delta\eta}(p)\mathcal{W}_{\rho\gamma}(p)\\ &= \frac{1}{2}(2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta})\sigma^{\mathrm{b}}_{\eta\rho}\mathcal{W}_{\beta\alpha}(p)\mathcal{W}^{\dagger}_{\delta\eta}(p)\mathcal{W}_{\rho\gamma}(p) = \sigma^{\mathrm{b}}_{\eta\rho}\mathcal{W}_{\rho\eta}(p) = n^{\mathrm{b}}(p) \qquad \because\ \mathcal{W}^{\dagger}\mathcal{W} = \mathcal{I}\end{aligned}$$

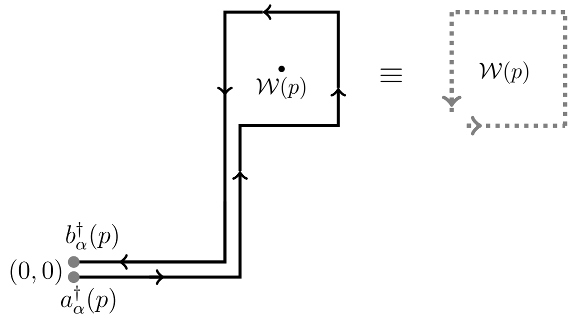


Figure 6.2: SU(2) prepotential operators in the dual formulation: The two ends of the plaquette flux operator $\mathcal{W}(p)$ are associated with two doublets of the harmonic oscillators at the origin (0, 0) [61, 78, 79, 139]. Under gauge transformations at the origin, $\left(a^\dagger_\alpha(p), b^\dagger_\beta(p)\right)$ transform as SU(2) doublets. The dotted plaquette on the right-hand side is a compact way to represent the plaquette holonomy $\mathcal{W}(p)$.

A. SU(2) Prepotential Operators

It is extremely convenient to use the prepotential [61, 78, 79] representation for the dual electric potential on the plaquette loops to construct the electric loop (Appendix C) as well the magnetic loop (Appendix D) basis. This simplification is illustrated in Figure 6.2. A further advantage is that this simple procedure can be directly generalized to all SU(N). We write the SU(2) dual plaquette loop electric potentials on any plaquette p satisfying (6.8) as [4]

$$\mathcal{E}^{\rm a}_{+}(p) \equiv a^\dagger(p)\frac{\sigma^{\rm a}}{2}a(p); \qquad \mathcal{E}^{\rm a}_{-}(p) \equiv -b(p)\frac{\sigma^{\rm a}}{2}b^\dagger(p). \tag{6.17}$$

In (6.17), $a^\dagger_\alpha(p)$ and $b^\dagger_\alpha(p)$ are the two mutually commuting SU(2) doublets of harmonic oscillator creation operators on every plaquette loop. The standard commutation relations are

$$\left[a_\alpha(p), a^\dagger_\beta(p')\right] = \delta_{pp'}\delta_{\alpha\beta}, \qquad \left[b_\alpha(p), b^\dagger_\beta(p')\right] = \delta_{pp'}\delta_{\alpha\beta}. \tag{6.18}$$

Using (6.18), it is easy to check that the representation (6.17) satisfies (6.8). The constraints (6.9) imply that

$$N(p) \equiv a^\dagger(p)\cdot a(p) = b^\dagger(p)\cdot b(p). \tag{6.19}$$

The plaquette holonomy in this representation is [61, 139]

$$\mathcal{W}_{\alpha\beta}(p) = F(N)\left[b^\dagger_\alpha(p)a^\dagger_\beta(p) + \tilde{b}_\alpha\tilde{a}_\beta\right]F(N) \tag{6.20}$$

[4] In defining $\mathcal{E}^a_{-}(p)$ we have used the fact that like (σ^a), their transpose with a negative sign $(-\tilde{\sigma}^a)$ also satisfies the same SU(2) Lie algebra.

In (6.20) $F(N) \equiv \frac{1}{\sqrt{(N(p)+1)}}$ is the normalization factor and $\tilde{x}_\alpha \equiv \epsilon_{\alpha\beta} x_\beta$. The harmonic oscillator representation (6.17) implies that $a^\dagger_\alpha$ and $b^\dagger_\alpha$ transform like doublet from right and anti-doublet from left respectively on every plaquette (p):

$$\left[\mathcal{E}^{\rm a}_+(p), a^\dagger_\alpha(p)\right] = -\left(a^\dagger(p)\,\frac{\sigma^{\rm a}}{2}\right)_\alpha, \qquad \left[\mathcal{E}^{\rm a}_-(p), b^\dagger_\alpha(p)\right] = \left(\frac{\sigma^{\rm a}}{2}\,b^\dagger(p)\right)_\alpha. \tag{6.21}$$

The strong coupling vacuum on every plaquette in the dual formulation $|0\rangle_p$ satisfies:

$$\mathcal{E}^{\rm a}_\pm(p)\;|0\rangle_p = 0, \quad \forall p. \tag{6.22}$$

This is equivalent to demanding

$$a_\alpha(p)|0\rangle_p = 0, \quad b_\alpha(p)|0\rangle_p = 0. \tag{6.23}$$

The relations (6.21) and (6.23) will be useful to study the action of SU(2) disorder operators on the magnetic basis discussed below. Note that under SU(2) gauge transformations (6.10) with $\Lambda(0,0)$ at the origin (see Figure 6.1-b) these oscillators transform doublets:

$$a^\dagger_\alpha(p) \rightarrow \;a^\dagger_\beta(p)\;\Lambda_{\beta\alpha}(0,0), \qquad b^\dagger_\alpha(p) \rightarrow \quad \Lambda^\dagger_{\alpha\beta}(0,0)\;b^\dagger_\beta(p) \qquad \forall p. \tag{6.24}$$

These relations are useful to construct the gauge invariant operators in the magnetic basis constructed in the next section.

B. SU(2) Magnetic Basis

The physical meaning of the operators $\Sigma^\pm_\theta(p)$ is simple. The non-Abelian electric scalar potentials $\mathcal{E}^{\rm a}_\pm(p)$ are conjugate to the magnetic flux operators $\mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\beta}(p)$. They satisfy the canonical commutation relations (6.7). Therefore, the gauge invariant vortex operator operator $\Sigma^\pm_\theta(p)$ acting on the magnetic basis on a plaquette changes the magnetic flux on it continuously as a function of θ in (6.35). To see this explicitly, we first construct the SU(2) magnetic basis. We note that

$$[\mathcal{W}_{\alpha\beta}(p), \mathcal{W}_{\gamma\delta}(p')] = 0, \;\; \forall\; p, p'.$$

Therefore we can diagonalize all 4 operators $(\mathcal{W}_{11}(p), \mathcal{W}_{12}(p), \mathcal{W}_{21}(p), \mathcal{W}_{22}(p))$ simultaneously on every plaquette. The common eigenstates $|Z(p)\rangle \equiv |z_1(p), z_2(p)\rangle$ satisfy

$$\mathcal{W}_{\alpha\beta}(p)\;|Z(p)\rangle = Z_{\alpha\beta}(p)\;|Z(p)\rangle, \qquad \alpha,\beta = 1,2. \tag{6.25}$$

In (6.25) the SU(2) matrix on the plaquette p is

$$Z = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}, \qquad |z_1|^2 + |z_2|^2 = 1. \tag{6.26}$$

The SU(2) Z matrices can also be written in the SU(2) angle-axis representation

$$Z = e^{i\,\omega\,\hat{n}^{\mathrm{a}}\,\frac{\sigma^{\mathrm{a}}}{2}}. \tag{6.27}$$

The two SU(2) representations (6.26) and (6.27) are related by

$$z_1 = \cos\left(\frac{\omega}{2}\right) + i\hat{n}^3 \sin\left(\frac{\omega}{2}\right), \qquad z_2 = (\hat{n}^2 + i\hat{n}^1)\sin\left(\frac{\omega}{2}\right).$$

We now construct $|Z(p)\rangle$ and show that on this basis the vortex operator $\Sigma_\theta^\pm(p)$ act as the shift operators for the plaquette magnetic fluxes. The magnetic eigenstates $|Z(p)\rangle$ can be explicitly constructed in terms of SU(2) prepotential operators [61, 78, 79, 139] (see Appendix D):

$$|Z(p)\rangle = \sum_{j(p)=0}^{\infty} \sqrt{d(j(p))}\, \frac{\left(a^\dagger(p)\ Z(p)\ b^\dagger(p)\right)^{2j(p)}}{(2j(p))!}\ |0\rangle_p. \tag{6.28}$$

In (6.28) $d(j) \equiv (2j+1)$ is the dimension of the j representation and $(a^\dagger\, Z\, b^\dagger) \equiv \sum_{\alpha,\beta=1}^{2} \left(a_\alpha^\dagger\, Z_{\alpha\beta}\, b_\beta^\dagger\right)$. From now onwards we will ignore the plaquette index p on all the operators and the states as they are all defined on the lattice plaquettes. The magnetic eigenstates (6.28) have simple SU(2) gauge transformation properties

$$|Z\rangle \to |\Lambda Z \Lambda^\dagger\rangle, \quad \Lambda \equiv \Lambda(0,0). \tag{6.29}$$

The transformations (6.29) are clear from (6.24) and (6.28). In the angle axis representation (6.27) the gauge transformations (6.29) take simpler form

$$\omega(p) \to \omega(p), \quad \hat{n}(p) \ \to\ \Lambda\,\hat{n}(p)\,\Lambda^\dagger, \qquad \Lambda \equiv \Lambda(0,0). \tag{6.30}$$

Thus $\omega(p)$, $\forall\ p$ are gauge invariant angles and $\hat{n}(p)$ $\forall p$ transform globally like SU(2) adjoint vectors. The eigenvalues of the plaquette magnetic field operators are:

$$\mathrm{Tr}\left(\mathcal{W}^{[j=\frac{1}{2}]}\right)\ |Z(\omega,\hat{n})\rangle = 2\cos\left(\frac{\omega}{2}\right)\ |Z(\omega,\hat{n})\rangle. \tag{6.31}$$

Now we evaluate the action of disorder operator using the prepotential relations

$$\Sigma_\theta^+ a_\alpha^\dagger \Sigma_\theta^- = \left(a^\dagger e^{\frac{i}{2}\theta\hat{n}^{\mathrm{a}}\sigma^{\mathrm{a}}}\right)_\alpha, \qquad \Sigma_\theta^- b_\alpha^\dagger \Sigma_\theta^+ = \left(e^{-\frac{i}{2}\theta\hat{n}^{\mathrm{a}}\sigma^{\mathrm{a}}} b^\dagger\right)_\alpha. \tag{6.32}$$

The relations (6.32) can be easily established using (6.15) and the prepotential representation of $\mathcal{E}_{\pm}(p)$ in (6.17).

$$\begin{aligned}\Sigma^{+}_{\theta}|Z(\omega,\hat{n})\rangle &= |e^{\,\frac{i}{2}\theta\hat{n}^{a}\sigma^{a}}Z(\omega,\hat{n})\rangle = |Z(\omega+\theta,\hat{n})\rangle \\ \Sigma^{-}_{\theta}|Z(\omega,\hat{n})\rangle &= |Z(\omega,\hat{n})e^{-\frac{i}{2}\theta\hat{n}^{a}\sigma^{a}}\rangle = |Z(\omega-\theta,\hat{n})\rangle\end{aligned} \tag{6.33}$$

Thus the SU(2) plaquette disorder operator $\Sigma^{\pm}_{\theta}$ translates the plaquette magnetic fluxes. This is precisely dual to the action of the Wilson loop operators which translate the SU(2) loop electric fluxes as shown in Appendix C (see eqn (C.10) and Figure C.1).

C. SU(2) Order-Disorder Algebra

The dual canonical commutation relations (6.7) involving magnetic plaquette flux operators $\mathcal{W}(p)$ and their conjugate electric scalar potential $\mathcal{E}(p)$ immediately lead to the SU(2) order-disorder algebra:

$$\begin{aligned}\Sigma^{+}_{\theta}(p)\,\mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\beta}(p)\,\Sigma^{-}_{\theta}(p) &= D^{[j=\frac{1}{2}]}_{\alpha\gamma}(\hat{n},\theta)\,\mathcal{W}^{[j=\frac{1}{2}]}_{\gamma\beta}(p) \\ \Sigma^{-}_{\theta}(p)\,\mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\beta}(p)\,\Sigma^{+}_{\theta}(p) &= \mathcal{W}^{[j=\frac{1}{2}]}_{\alpha\gamma}(p)\,D^{[j=\frac{1}{2}]}_{\gamma\beta}(\hat{n},\theta).\end{aligned} \tag{6.34}$$

In (6.34) the Wigner matrix $D^{[j=\frac{1}{2}]} \equiv e^{i\,\hat{n}^{a}\cdot\,\sigma^{a}\,\frac{\theta}{2}}$ are the rotation matrix in $j=\frac{1}{2}$ representation around the magnetic axis $\hat{n}(p)$ defined through the plaquette loops $\mathcal{W}(p)$. In any higher $[j]$ representation, we can write:

$$\mathcal{W}^{[j]}_{\alpha\beta} = \mathcal{W}^{[j=1/2]}_{\{\alpha_1\beta_1}\mathcal{W}^{[j=1/2]}_{\alpha_2\beta_2}\cdots\mathcal{W}^{[j=1/2]}_{\alpha_{2j}\beta_{2j}\}}$$

with all the α (and therefore β) indices are completely symmetrized. Inserting the disorder operators (Σ) and their inverses ($\Sigma^{\dagger}$) in the middle, we get the SU(2) order-disorder algebra relation in j representation.

$$\begin{aligned}\Sigma^{+}_{\theta}(p)\;\mathcal{W}^{[j]}_{\alpha\beta}(p)\;\Sigma^{-}_{\theta}(p) &= D^{[j]}_{\alpha\gamma}(\hat{n},\theta)\;\mathcal{W}^{[j]}_{\gamma\beta}(p), \\ \Sigma^{-}_{\theta}(p)\;\mathcal{W}^{[j]}_{\alpha\beta}(p)\;\Sigma^{+}_{\theta}(p) &= \mathcal{W}^{[j]}_{\alpha\gamma}(p)\;D^{[j]}_{\gamma\beta}(\hat{n},\theta).\end{aligned} \tag{6.35}$$

In the special case when the rotations are restricted to the centre Z_2 of the SU(2) group then $\theta = 0$ or 2π in (6.35) and we recover the 't Hooft Wilson order-disorder algebra with $D^{[j]}_{\alpha\beta}(\theta=2\pi) = (-1)^{2j}\,\delta_{\alpha\beta}$.

$$\Sigma^{\pm}_{\theta=2\pi}\,\mathcal{W}^{[j]}_{\alpha\beta} = (-1)^{2j}\,\mathcal{W}^{[j]}_{\alpha\beta}\,\Sigma^{\pm}_{\theta=2\pi}. \tag{6.36}$$

In (6.36), $(-1)^{2j}$ is the n-ality of the j representation. We thus recover the standard Wilson-'t Hooft loop Z_2 algebra [94–97,154,155,157,158] for SU(2) at $\theta = 2\pi$. The operator $\Sigma_{2\pi} \equiv \Sigma^+_{2\pi} = \Sigma^-_{2\pi}$ is the SU(2) 't Hooft operator.

SU(3) Disorder Operator

In this section, we construct the disorder operator for SU(3) lattice gauge theory before going to SU(N) gauge group. As in the previous SU(2) case, they are the SU(3) magnetic vortex creation-annihilation operators and are expected to magnetically disorder the weak coupling ground state [154, 159–165]. The SU(3) plaquette magnetic flux operators can be written as

$$\mathcal{W}^{[\mathrm{p}=1\ \mathrm{q}=1]}(p) = \exp i \left(\hat{n}(p) \cdot \vec{\lambda}\ \ \omega(p)\right). \tag{6.37}$$

In (6.37) $\lambda^{\mathrm{a}}(\mathrm{a} = 1, \cdots, 8)$ are the Gell-Mann matrices. We can also use the angle-axis representation [166] to write:

$$\mathcal{W}^{[\mathrm{p}=1\ \mathrm{q}=1]}(p) \equiv A\mathcal{I} + B\,\vec{n}\cdot\vec{\lambda} + C\,\vec{n}\star\vec{n}\cdot\vec{\lambda}. \tag{6.38}$$

In (6.38) $(\vec{n}\star\vec{n})^{\mathrm{a}} \equiv d^{\mathrm{abc}}\,\vec{n}^{\mathrm{b}}\,\vec{n}^{\mathrm{c}}$ defines the second independent vector with the help of the SU(3) symmetric tensors d^{abc}. Instead of following the standard polar decomposition (6.38), it is more convenient for us to construct the two independent SU(3) axes operators as [5]

$$\vec{n}^{\mathrm{a}}_{[1]}(p) = \mathrm{Tr}\lambda^{\mathrm{a}}\left(\mathcal{W}^{[1,1]}(p) + \mathcal{W}^{\dagger[1,1]}(p)\right), \tag{6.39a}$$

$$\vec{n}^{\mathrm{a}}_{[2]}(p) = \sqrt{3}\, d^{\mathrm{abc}}\vec{n}^{\mathrm{b}}_{[1]}(p)\vec{n}^{\mathrm{c}}_{[1]}(p). \tag{6.39b}$$

Note that $\vec{n}^{\mathrm{a}}_{[1]}(p), \vec{n}^{\mathrm{a}}_{[2]}(p)$ are real. Under SU(3) gauge transformations (6.10) the above two operators transform as:

$$\vec{n}^{\mathrm{a}}_{[1]}(p) \to R^{\mathrm{ab}}(\Lambda)\,\vec{n}^{\mathrm{b}}_{[1]}(p), \qquad \vec{n}^{\mathrm{a}}_{[2]}(p) \to R^{\mathrm{ab}}(\Lambda)\,\vec{n}^{\mathrm{b}}_{[2]}(p). \tag{6.40}$$

In (6.40) $R^{\mathrm{ab}}(\Lambda) = \frac{1}{2}\mathrm{Tr}(\lambda^{\mathrm{a}}\Lambda^{\dagger}\lambda^{\mathrm{b}}\Lambda)$ and $\Lambda \equiv \Lambda(0,0)$. These two axes are linearly independent. It can be shown that in SU(3) case there exist only two independent axes as the third axis defined using another d^{abc} is the first axis $\vec{n}_{[1]}$ [6]:

$$f^{\mathrm{abc}}\vec{n}^{\mathrm{b}}_{[2]}(p)\vec{n}^{\mathrm{c}}_{[1]}(p) = 0, \qquad d^{\mathrm{abc}}\vec{n}^{\mathrm{b}}_{[2]}(p)\vec{n}^{\mathrm{c}}_{[1]}(p) = \frac{1}{\sqrt{3}}(\vec{n}^{\mathrm{b}}_{[1]}(p)\vec{n}^{\mathrm{b}}_{[1]}(p))\ \vec{n}^{\mathrm{a}}_{[1]}(p).$$

---

[5] The two definitions for $\vec{n}_{[1]}(p)$ and $\vec{n}_{[2]}(p)$ in (6.39a) and (6.39b) respectively are easily generalizable to SU(N) case discussed in the next section.

[6] We have used the following two identities

1. $(f^{\mathrm{abe}}d^{\mathrm{dce}} + f^{\mathrm{ace}}d^{\mathrm{bde}} + f^{\mathrm{ade}}d^{\mathrm{bce}}) = 0$
2. $(d^{\mathrm{abe}}d^{\mathrm{dce}} + d^{\mathrm{ace}}d^{\mathrm{bde}} + d^{\mathrm{ade}}d^{\mathrm{bce}}) = \frac{1}{3}(\delta^{\mathrm{ab}}\delta^{\mathrm{cd}} + \delta^{\mathrm{ac}}\delta^{\mathrm{bd}} + \delta^{\mathrm{bc}}\delta^{\mathrm{ad}})$

Now we define the SU(3) disorder operators which translate these two gauge invariant magnetic fluxes:

$$\begin{aligned}\Sigma^{+}_{\theta_1,\theta_2}(p) \equiv & \exp\left(i\sum_{h=1}^{2}\theta_h(p)\hat{n}^{\mathrm{a}}_{[h]}\right)\mathcal{E}^{\mathrm{a}}_{+}(p), \\ \Sigma^{-}_{\theta_1,\theta_2}(p) \equiv & \exp\left(i\sum_{h=1}^{2}\theta_h(p)\hat{n}^{\mathrm{a}}_{[h]}\right)\mathcal{E}^{\mathrm{a}}_{-}(p)\end{aligned} \tag{6.41}$$

In (6.41) $(\theta_1, \theta_2) \equiv (\theta_1(p), \theta_2(p))$ are the external angular parameters characterizing the SU(3) disorder operator. Like in the SU(2) case, the two operators in (6.41) are unitary and Hermitian conjugate of each other

$$\Sigma^{+}_{\theta_1,\theta_2}(p)\,\Sigma^{-}_{\theta_1,\theta_2}(p) = \mathcal{I} = \Sigma^{-}_{\theta_1,\theta_2}(p)\,\Sigma^{+}_{\theta_1,\theta_2}(p). \tag{6.42}$$

Like SU(2) case this can also be proved using the properties of the SU(3) λ matrices.

A. SU(3) Prepotential Operators

The SU(3) prepotential operators on plaquettes are defined as

$$\mathcal{E}^{\mathrm{a}}_{+} \equiv \sum_{h=1}^{2} a^{\dagger}[h]\,\frac{\lambda^{\mathrm{a}}}{2}\,a[h], \qquad \mathcal{E}^{\mathrm{a}}_{-} \equiv -\sum_{h=1}^{2} b[h]\,\frac{\lambda^{\mathrm{a}}}{2}\,b^{\dagger}[h]. \tag{6.43}$$

In (6.43), $(a^{\dagger}_{\alpha}[h], a_{\alpha}[h])$ and $(b^{\dagger}_{\alpha}[h], b_{\alpha}[h])$ where $\alpha = 1, 2, 3; h = 1, 2$ are the mutually independent SU(3) triplets of harmonic oscillator creation-annihilation operators on every plaquette [7]. They are attached to the initial and the end points of the plaquette loops (see Figure 6.1-b). The summation over $[h] = 1, 2$ is over the rank of the group. As all operators are defined on plaquettes, we suppress the plaquette index 'p' throughout this section. The harmonic oscillator commutation relations and (6.43) imply that $a^{\dagger}_{\alpha}[h]$ and $b^{\dagger}_{\alpha}[h]$ transform like triplets from right and anti-triplets from left respectively on every plaquette (p): transformations:

$$\left[\mathcal{E}^{\mathrm{a}}_{+}, a^{\dagger}_{\alpha}[h]\right] = \left(a^{\dagger}[h]\,\frac{\lambda^{\mathrm{a}}}{2}\right)_{\alpha}, \qquad \left[\mathcal{E}^{\mathrm{a}}_{-}, b^{\dagger}_{\alpha}[h]\right] = -\left(\frac{\lambda^{\mathrm{a}}}{2}\,b^{\dagger}[h]\right)_{\alpha}, \qquad h = 1, 2. \tag{6.44}$$

[7] We are ignoring the SU(3) multiplicity problem here as the aim in this work is to construct the SU(3) magnetic eigenstates and not to worry about SU(3) multiplicities. One can trivially replace all SU(3) prepotentials by SU(3) irreducible prepotentials [61, 78, 79, 138–140] at the end without changing any results of this section. The same strategy will be adapted in the next SU(N) section to keep the discussion simple.

Like in SU(2) case (6.24), the SU(3) gauge transformations (6.10) with $\Lambda(0,0)$ at the orgin (see Figure 6.1-b) the SU(3) oscillators on every plaquette transform as SU(3) triplets:

$$a^\dagger_\alpha[h] \rightarrow \; a^\dagger_\beta[h]\;\Lambda_{\beta\alpha}(0,0), \qquad b^\dagger_\alpha[h] \rightarrow \quad \Lambda^\dagger_{\alpha\beta}(0,0)\; b^\dagger_\beta[h], \qquad h = 1,2. \tag{6.45}$$

These relations are again useful for the gauge covariant parametrization of the SU(3) magnetic basis in the angle axis representation and is discussed in the next section. The SU(3) strong coupling vacuum in the dual description $|0\rangle$ satisfies

$$a_\alpha[h]\;|0\rangle_p = 0, \quad b_\alpha[h]\;|0\rangle_p = 0, \qquad h = 1,2. \tag{6.46}$$

This strong coupling vacuum state $|0\rangle_p \equiv |0\rangle$ will be used to construct the SU(3) magnetic basis in the next section.

B. SU(3) Magnetic Basis

We now show that $\Sigma^{\pm}_{\theta_1,\theta_2}$ operating on the SU(3) plaquette magnetic basis acts like a translation operator for the two gauge invariant magnetic fields. As shown in Appendix D, the SU(3) magnetic basis can be written in terms of SU(3) pre-potentials [61, 78, 79, 139] as:

$$|Z\rangle = \sum_{\mathrm{p,q}=0}^{\infty} \sqrt{d(\mathrm{p,q})}\frac{(a^\dagger[1]Zb^\dagger[1])^{\mathrm{p}}}{\mathrm{p}!}\frac{(a^\dagger[2]Zb^\dagger[2])^{\mathrm{q}}}{\mathrm{q}!}\;|0\rangle \tag{6.47}$$

In the above equation, the plaquette index has been suppressed and

$$d(\mathrm{p},\;\mathrm{q}) = \frac{1}{2}(\mathrm{p}+1)(\mathrm{q}+1)(\mathrm{p}+\mathrm{q}+2),$$

is the dimension of the [p, q] representation of SU(3) [167], $Z_{\alpha\beta}$ are the elements of SU(3) matrix and correspond to the eigenvalues of $\mathcal{W}^{[\mathrm{p}=1,\mathrm{q}=1]}_{\alpha\beta}(p)$ and we have ignored plaquette index p in (6.47). In the axis-angle representation Z can be written as [8]

$$Z(p) = Z(\omega_1,\omega_2) = \exp i\big(\omega_1\hat{n}^{\mathrm{a}}_{[2]} + \omega_2\hat{n}^{\mathrm{a}}_{[2]}\big)\frac{\lambda^{\mathrm{a}}}{2}. \tag{6.48}$$

[8]Advantage of this representation is that it has the following property:

$$Z(\omega_1,\omega_2)Z(\theta_1,\theta_2) = e^{\left(i(\omega_1\hat{n}^{\mathrm{a}}_{[1]}+\omega_2\hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}\right)}e^{\left(i(\theta_2\hat{n}^{\mathrm{b}}_{[1]}+\theta_2\hat{n}^{\mathrm{b}}_{[2]})\frac{\lambda^{\mathrm{b}}}{2}\right)}$$

$$\because e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}, \quad [\lambda^{\mathrm{a}},\lambda^{\mathrm{b}}] = 2if^{\mathrm{abc}}\lambda^{\mathrm{c}}, \quad f^{\mathrm{abc}}\hat{n}^{\mathrm{b}}_{[h]}\hat{n}^{\mathrm{c}}_{[h']} = 0, \qquad h,h' = 1,2$$

$$Z(\omega_1,\omega_2)Z(\theta_1,\theta_2) = e^{\left(i((\omega_1+\theta_1)\hat{n}^{\mathrm{a}}_{[1]}+(\omega_1+\theta_2)\hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}\right)} = Z(\omega_1+\theta_1,\;\omega_2+\theta_2)$$

Which we will use to show the translation of two gauge invariant angles through the action of the disorder operator.

In (6.48) we have labeled the SU(3) group manifold by $Z(\omega_1, \omega_2) \equiv Z(\hat{n}_{[1]}, \hat{n}_{[2]}; \omega_1, \omega_2)$. The two axes $(\hat{n}_{[1]}, \hat{n}_{[2]})$ are suppressed for the notational simplicity. Under SU(3) gauge transformations at the origin (6.10)

$$|Z\rangle \rightarrow |\Lambda\, Z\, \Lambda^{\dagger}\rangle, \qquad \Lambda \equiv \Lambda(0,0). \tag{6.49}$$

We have used (6.45) and the defining equations (6.48) to obtain the above covariant transformations. The gauge transformations (6.49) show that

$$\omega_h \rightarrow \omega_h, \qquad \hat{n}_{[h]} \rightarrow \Lambda\ \hat{n}_{[h]}\ \Lambda^{\dagger} \qquad h = 1, 2. \tag{6.50}$$

Thus (ω_1, ω_2) are the gauge invariant angles and the two axes $\hat{n}_{[h]} \equiv \sum_{\mathrm{a}=1}^{8} \hat{n}^{\mathrm{a}}_{[h]} \lambda^{\mathrm{a}}$ transform like the adjoint vectors on every plaquette.

In order to evaluate the action of the disorder operator on this magnetic basis we first write down the following equations, which can be easily established using the commutation relations in (6.44)

$$\Sigma^{+}_{\theta_1,\theta_2}\, a^{\dagger}_{\alpha}[h]\, \Sigma^{+\dagger}_{\theta_1,\theta_2} = \left(a^{\dagger}[h]\ e^{i\left(\theta_1 \hat{n}^{\mathrm{a}}_{[2]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]}\right)\frac{\lambda^{\mathrm{a}}}{2}}\right)_{\alpha}, \tag{6.51a}$$

$$\Sigma^{-}_{\theta_1,\theta_2}\, b^{\dagger}_{\alpha}[h]\, \Sigma^{-\dagger}_{\theta_1,\theta_2} = \left(e^{-i\left(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]}\right)\frac{\lambda^{\mathrm{a}}}{2}}\ b^{\dagger}[h]\right)_{\alpha} \tag{6.51b}$$

Using the above equations we can easily prove that

$$\begin{aligned}
\Sigma^{+}_{\theta_1,\theta_2}|Z(\omega_1,\omega_2)\rangle &= |e^{\left(i(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}\right)} Z(\omega_1,\omega_2)\rangle = |Z(\omega_1+\theta_1, \omega_2+\theta_2)\rangle, \\
\Sigma^{-}_{\theta_1,\theta_2}|Z(\omega_1,\omega_2)\rangle &= |Z(\omega_1,\omega_2) e^{\left(-i(\theta_1 \hat{n}^{\mathrm{a}}_{[1]} + \theta_2 \hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}\right)}\rangle = |Z(\omega_1-\theta_1, \omega_2-\theta_2)\rangle
\end{aligned}$$

or,

$$\Sigma^{\pm}_{\theta_1,\theta_2}\ |Z(\omega_1,\omega_2)\rangle =\ |Z(\omega_1 \pm \theta_1,\ \omega_2 \pm \theta_2)\rangle \tag{6.52}$$

Therefore the disorder operator in (6.41) translate two gauge invariant angles. We can thus interpret them as creation-annihilation operators for SU(3) magnetic vortices.

C. SU(3) Order-Disorder Algebra

The SU(3) order-disorder algebra is

$$\begin{aligned}
\Sigma^{+}_{[\vec{\theta}]}(p)\ \mathcal{W}^{[\mathrm{p},\,\mathrm{q}]}_{\alpha\beta}(p)\ \Sigma^{+\dagger}_{[\vec{\theta}]}(p) &= D^{[\mathrm{p},\,\mathrm{q}]}_{\alpha\gamma}(\vec{\theta})\, \mathcal{W}^{[\mathrm{p},\,\mathrm{q}]}_{\gamma\beta}(p) \\
\Sigma^{-}_{[\vec{\theta}]}(p) \mathcal{W}^{[\mathrm{p},\,\mathrm{q}]}_{\alpha\beta}(p) \Sigma^{-\dagger}_{[\vec{\theta}]}(p) &= \mathcal{W}^{[\mathrm{p},\,\mathrm{q}]}_{\alpha\gamma}(p)\, D^{[\mathrm{p},\,\mathrm{q}]}_{\gamma\beta}(\vec{\theta})
\end{aligned} \tag{6.53}$$

In (6.53), $D^{[\mathrm{p,q}]}(\theta_1,\theta_2) \equiv \exp\Big(i(\theta_1\hat{n}^{\mathrm{a}}_{[1]}+\theta_2\hat{n}^{\mathrm{a}}_{[2]})\frac{\lambda^{\mathrm{a}}}{2}\Big)$ is SU(3) Wigner D-matrix in the [p, q] representation. Like is SU(2) case, we have used the dual canonical commutation relations (6.7) to obtain the SU(3) order-disorder algebra in (6.53).

SU(N) Disorder Operator

We now use the SU(N) dual electric scalar potentials $\mathcal{E}(p)$ in (6.4) to define the SU(N) disorder operator

$$\Sigma^{\pm}_{[\theta_1\theta_2\cdots\theta_{\mathrm{N}-1}]}(p) = \exp\ i\Big\{\ \vec{\theta}(p)\cdot\ \vec{\mathcal{E}}_{\pm}(p)\Big\}. \tag{6.54}$$

In (6.54) $[\theta_1,\theta_2,\cdots,\theta_{\mathrm{N}-1}] \equiv [\theta_1(p),\theta_2(p),\cdots,\theta_{\mathrm{N}-1}(p)]$ are the (N − 1) external angular parameters characterizing the SU(N) disorder operator on the plaquette (p) and

$$\vec{\theta}(p) \equiv \sum_{h=1}^{(\mathrm{N}-1)} \theta_h(p)\hat{n}_{[h]}(p). \tag{6.55}$$

The invariance (6.10) demands that the operator $\vec{\theta}(p)$ in (6.54) is the most general vector operator constructed out of magnetic flux operator $\mathcal{W}_{\alpha\beta}(p)$. In other words, they depend on the (N − 1) directions of the SU(N) magnetic fields. In SU(2) and SU(3) cases in the previous sections we have already constructed one and two independent axes respectively using the plaquette magnetic flux operators. In the same way we now iteratively define the (N − 1) linearly independent "SU(N) magnetic axes" using the SU(N) symmetric structure constants d^{abc} as follows:

$$\vec{n}^{\mathrm{a}}_{[h+1]}(p) \equiv d^{\mathrm{abc}}\ \vec{n}^{b}_{[h]}(p)\ \vec{n}^{c}_{[1]}(p)\quad h=1,2,\ldots,\mathrm{N}-2. \tag{6.56}$$

The first magnetic axis is defined as $\vec{n}^{\mathrm{a}}_{[1]}(p) \equiv \mathrm{Tr}(\Lambda^{\mathrm{a}}\left(\mathcal{W}+\mathcal{W}^{\dagger}\right))$ where $\Lambda^{\mathrm{a}}(\mathrm{a}=1,2,\cdots,(\mathrm{N}^2-1))$ are the SU(N) fundamental representation matrices. The iterative procedure ends as [9] $\vec{n}^{\mathrm{a}}_{[\mathrm{N}]} \equiv d^{\mathrm{abc}}\ \vec{n}^{\mathrm{b}}_{[\mathrm{N}-1]}(p)\ \vec{n}^{\mathrm{c}}_{[1]}(p) = \vec{n}^{\mathrm{a}}_{[1]}(p)$. The (N − 1) SU(N) magnetic field operators $\vec{n}^{\mathrm{a}}_{[h]};\ h=1,2,\cdots,(\mathrm{N}-1)$ are Hermitian as the symmetric structure constants d^{abc} are always real. Under gauge transformation (6.10), these axes transform as vectors

$$\vec{n}^{\mathrm{a}}_{[h]}(p) \rightarrow\ R^{\mathrm{ab}}(\Lambda)\ \vec{n}^{\mathrm{b}}_{[h]}(p),\quad h=1,2,\cdots,(\mathrm{N}-1). \tag{6.57}$$

The disorder operator is invariant under the gauge transformations (6.10) as $\vec{\theta}(p)$ and the dual electric potentials $\vec{\mathcal{E}}(p)$ both transform as vectors. As in the case of SU(2) (see (6.16)) and SU(3) (see (6.42)), $\Sigma^{+}_{[\vec{\theta}]}(p)$ and $\Sigma^{-}_{[\vec{\theta}]}(p)$ are not independent and satisfy

$$\Sigma^{+}_{[\vec{\theta}]}(p)\ \Sigma^{-}_{[\vec{\theta}]}(p)\ =\ \mathcal{I}\ =\ \Sigma^{-}_{[\vec{\theta}]}(p)\ \Sigma^{+}_{[\vec{\theta}]}(p). \tag{6.58}$$

[9] We have used the property: $(d^{\mathrm{abe}}d^{\mathrm{cde}}+d^{\mathrm{ace}}d^{\mathrm{bde}}+d^{\mathrm{ade}}d^{\mathrm{bce}}) = \frac{1}{3}(\delta^{\mathrm{ab}}\delta^{\mathrm{cd}}+\delta^{\mathrm{ac}}\delta^{\mathrm{bd}}+\delta^{\mathrm{bc}}\delta^{\mathrm{ad}})$ for SU(3). It can be similarly generalized to SU(N) with (N − 1) d structure functions.

Here $\mathcal{I}$ is unity operator in the physical Hilbert space. The relations (6.58) follow from the parallel transport relating the two electric scalar potentials: $\mathcal{E}^{\rm a}_{-}(p) = -R^{\rm ab}(\mathcal{W}(p))\ \mathcal{E}^{\rm b}_{+}(p)$ and $\hat{n}^{\rm a}_{[h]}(p) = -R^{\rm ab}(\mathcal{W}(p))\ \hat{n}^{\rm b}_{[h]}(p)$; $h = 1, 2\cdots, (\mathrm{N}-1)$. We now briefly disscuss the SU(N) prepotential operators to be used in the Section 6.1 - B for the construction of SU(N) magnetic basis.

A. SU(N) Prepotential Operators

The SU(N) dual electric scalar potentials $\mathcal{E}^{a}(p)$ can be written in terms of the (N − 1) N-plets of harmonic oscillators at each of the two ends of the plaquette p. We define

$$\begin{aligned}
\mathcal{E}^{\rm a}_{+}(p) &= \frac{1}{2}\sum_{h=1}^{(\mathrm{N}-1)}\left[\sum_{\alpha,\,\beta=1}^{\mathrm{N}} a^{\dagger}_{\alpha}[h]\ (\Lambda^{\rm a})_{\alpha\beta}\, a_{\beta}[h]\right], \\
\mathcal{E}^{\rm a}_{-}(p) &= -\frac{1}{2}\sum_{h=1}^{(\mathrm{N}-1)}\left[\sum_{\alpha,\,\beta=1}^{\mathrm{N}} b_{\alpha}[h]\ (\Lambda^{\rm a})_{\alpha\beta}\, b^{\dagger}_{\beta}[h]\right].
\end{aligned} \tag{6.59}$$

In (6.59), we have introduced prepotential N-plets $(a_{\alpha}[h], a^{\dagger}_{\alpha}[h])$ and $(b_{\alpha}[h], b^{\dagger}_{\alpha}[h])$ for each of the (N − 1) fundamental representations of SU(N). They are denoted by $h = 1, 2, \cdots, (\mathrm{N}-1)$ and we have suppressed the additional plaquette index on the right hand side of (6.59) for convenience. The $\frac{\Lambda^{a}}{2}$ are the (N^2-1) SU(N) matrices in the fundamental representation. The harmonic oscillator commutation relations of the SU(N) prepotentials imply

$$\left[\mathcal{E}^{\rm a}_{+}[h],\ a^{\dagger}_{\alpha}[h']\right] = \delta_{h,h'}\,\frac{1}{2}\,a^{\dagger}_{\beta}[h]\ \Lambda^{\rm a}_{\beta\alpha}, \qquad \left[\mathcal{E}^{\rm a}_{+}[h],\ b^{\dagger}_{\alpha}[h']\right] = -\delta_{h,h'}\,\frac{1}{2}\,\Lambda^{\rm a}_{\alpha\beta}\ b^{\dagger}_{\beta}[h]. \tag{6.60}$$

We also note that under SU(N) gauge transformations (6.10) with $\Lambda \equiv \Lambda(0,0)$ (see Figure 6.1-b) these oscillators transform as

$$a^{\dagger}_{\alpha}[h] \to\ a^{\dagger}_{\beta}[h]\ \Lambda_{\beta\alpha}\,, \qquad b^{\dagger}_{\alpha}[h] \to\ \Lambda^{\dagger}_{\alpha\beta}\ b^{\dagger}_{\beta}[h]\,, \qquad \forall\ h = 1, 2, ..., (\mathrm{N}-1). \tag{6.61}$$

Like in SU(2) and SU(3) cases, the relations (6.60) and (6.61) will be useful in constructing the SU(N) magnetic basis in the next section.

B. SU(N) Magnetic Basis

In this section, we construct the SU(N) magnetic basis for all SU(N) and show that the disorder operators on a magnetic basis act as shift operators for the N-1 magnetic fields. The SU(N) magnetic basis has been constructed in Appendix D and is given by:

$$|Z\rangle = \sum_{[\vec{j}]=0}^{\infty} \sqrt{d(\vec{j})} \prod_{h=1}^{\mathrm{N}-1} \frac{1}{j_h!} \left(a^{\dagger}[h] Z b^{\dagger}[h]\right)^{2j_h} \ |0\rangle. \tag{6.62}$$

In (6.62) $\sqrt{d(\vec{j})}$ is the dimension of the SU(N) $[\vec{j}]$ ($\equiv (j_1, j_2, \cdots, j_{\mathrm{N}-1})$ representation. The SU(N) strong coupling vacuum $|0\rangle$ in the dual description on every plaquette satisfies

$$a_{\alpha}[h]\ |0\rangle = 0,\ b_{\alpha}[h]\ |0\rangle = 0, \quad h = 1, 2, \cdots, (\mathrm{N}-1). \tag{6.63}$$

Like in SU(2) and SU(3) cases we parameterize the SU(N) matrix $Z \equiv Z(p)$ in (6.62) on every plaquette p in the angle-axis representation as

$$Z = Z(\omega_1, \omega_2, \cdots, \omega_{\mathrm{N}-1}) = \exp i \left(\omega_h\, \hat{n}^{\mathrm{a}}_{[h]}\, \frac{\Lambda^{\mathrm{a}}}{2}\right). \tag{6.64}$$

In (6.64) the $(\mathrm{N}-1)$ linearly independent unit vectors are defined as

$$\vec{n}^{\mathrm{a}}_{[r+1]}(p) \equiv\ d^{\mathrm{abc}}\ \hat{n}^{\mathrm{b}}_{[r]}(p)\ \hat{n}^{\mathrm{c}}_{[1]}(p), \quad r = 1, 2, \ldots, \mathrm{N}-2. \tag{6.65}$$

We have again suppressed the $(\mathrm{N}-1)$ axes $\hat{n}^{a}_{[h]}$ in $Z(\omega_1, \omega_2, \cdots, \omega_{\mathrm{N}-1})$ for the notational simplicity.

In order to evaluate the action of disorder operators on the magnetic basis (6.62), we use (6.60) to obtain:

$$\Sigma^{+}_{[\vec{\theta}]}\ a^{\dagger}_{\alpha}[h]\ \Sigma^{+\dagger}_{[\vec{\theta}]} = \left(a^{\dagger}[h]\ e^{i\left(\theta_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}}\right)_{\alpha}, \qquad \Sigma^{-}_{[\vec{\theta}]}\ b^{\dagger}_{\alpha}[h]\ \Sigma^{-\dagger}_{[\vec{\theta}]} = \quad \left(e^{-i\left(\theta_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}}\ b^{\dagger}[h]\right)_{\alpha} \tag{6.66}$$

Therefore the action of disorder operators on the magnetic basis is given by

$$\Sigma^{+}_{[\vec{\theta}]}\ |Z\rangle = |e^{i\left(\theta_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}}\ Z\rangle, \qquad \Sigma^{-}_{[\vec{\theta}]}\ |Z\rangle =\ |Z\ e^{i\left(\theta_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}}\rangle \tag{6.67}$$

We now use axis angle-representation (6.64) to get

$$\begin{aligned} \Sigma^{+}_{[\vec{\theta}]}\ |Z(\omega_h)\rangle &= |e^{i\left(\theta_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}} e^{i\left(\omega_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}}\rangle =\ |Z(\omega_h + \theta_h)\rangle \\ \Sigma^{-}_{[\vec{\theta}]}\ |Z(\omega_h)\rangle &= |e^{i\left(\omega_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}} e^{-i\left(\theta_h \hat{n}^{\mathrm{a}}_{[h]}\right)\frac{\Lambda^{\mathrm{a}}}{2}}\rangle =\ |Z(\omega_h - \theta_h)\rangle \end{aligned} \tag{6.68}$$

Therefore the disorder operator on a plaquette p translates the $\mathrm{N}-1$ gauge invariant angles defining the SU(N) magnetic fluxes.

C. SU(N) Order-Disorder Algebra

Using the canonical commutation relations in the dual description (6.7) we get Similarly, the SU(N) order-disorder algebra is

$$\begin{aligned}\Sigma^{+}_{[\vec{\theta}]}(p)\, \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p)\, \Sigma^{-}_{[\vec{\theta}]}(p) &= D^{[\vec{j}]}_{\alpha\gamma}([\vec{\theta}])\, \mathcal{W}^{[\vec{j}]}_{\gamma\beta}(p)\\ \Sigma^{-}_{[\vec{\theta}]}(p)\, \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p)\, \Sigma^{+}_{[\vec{\theta}]}(p) &= \mathcal{W}^{[\vec{j}]}_{\alpha\gamma}(p) D^{[\vec{j}]}_{\gamma\beta}([\vec{\theta}]).\end{aligned} \tag{6.69}$$

In (6.69) the Wigner matrix $D^{[\vec{j}]}([\vec{\theta}])$ represent the SU(N) rotations around the magnetic axes $\hat{n}_{[h]}$ by θ_h with $h = 1, 2, \cdots, (\mathrm{N}-1)$.

Reduction to 't Hooft Algebra

In the special case when the rotations are in the center of SU(N) with $Z \in Z_{\mathrm{N}}$ and $Z^{\mathrm{N}} = 1$, we get

$$D^{[\vec{j}]}(Z) = (z)^{\eta[\vec{j}]}\ \mathcal{I}, \quad z^{\mathrm{N}} = 1$$

where $\mathcal{I}$ is $\mathrm{N}\times\mathrm{N}$ unit matrix and $\eta[\vec{j}]$ is the N-ality of the $[\vec{j}]$ representation. We thus get the 't Hooft Wilson order-disorder algebra [94–97, 154, 155, 157, 158].

$$\begin{aligned}\Sigma^{+}_{[Z_{\mathrm{N}}]}(p)\, \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p)\, \Sigma^{-}_{[Z_{\mathrm{N}}]}(p) &= (z)^{\eta[\vec{j}]}\, \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p)\\ \Sigma^{-}_{[Z_{\mathrm{N}}]}(p)\, \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p)\, \Sigma^{+}_{[Z_{\mathrm{N}}]}(p) &= (z)^{\eta[\vec{j}]}\, \mathcal{W}^{[\vec{j}]}_{\alpha\beta}(p).\end{aligned} \tag{6.70}$$

The SU(N) center elements in (6.70) are

$$z = e^{\frac{2\pi i m}{\mathrm{N}}},\ m = 0, 1, \cdots, (\mathrm{N}-1).$$

6.2 SU(N) Dirac Strings

The disorder operators defined in the previous section can also be written in terms of the Kogut-Susskind link holonomies and their electric fields using the exact duality transformations (6.4). As expected, these Disorder operators $\Sigma(p)$ are highly non-local operators in the original description but their physical action is essentially local. Using duality transformation relation (6.4) we write;

$$\Sigma^{+}_{[\theta_1\theta_2\cdots\theta_{\mathrm{N}-1}]}(m,n) = \exp\left\{ i\, \vec{\theta}^{\mathrm{a}}(m,n) \cdot \sum_{n'=n+1}^{\infty} R^{\mathrm{ab}}(\mathsf{S}(m,n;n'))E^{\mathrm{b}}_{-}(m,n';\hat{1}) \right\} \tag{6.71}$$

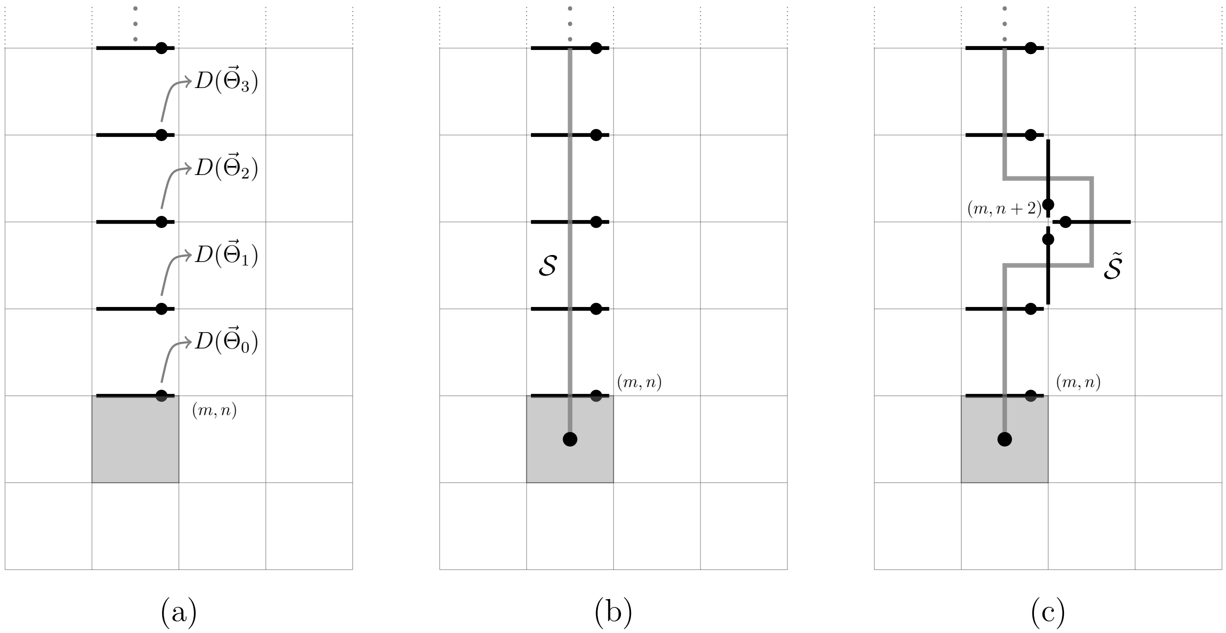


Figure 6.3: (a) The disorder operator $\Sigma^{+}_{[\vec{\theta}]}(m,n)$, defined in equation (6.71) rotates all horizontal links $U(m-1,n';\hat{1})$, $\forall\ n' \geq n$ around an axis $\vec{\Theta}(m,n')$ (for $n' = n, n+1, n+2, ..$ they are denoted by $\vec{\Theta}_0, \vec{\Theta}_1, \vec{\Theta}_2, ...$), (b) Invisible SU(N) Dirac string $\mathcal{S}$. The rotated links $\mathsf{l} \in \mathcal{S}$ are the dark horizontal links, (c) Shape of Dirac string can be deformed without affecting the endpoint or the location of the magnetic vortex. The SU(N) gauge transformations at site $(m, n+2)$ changes the shape of the Dirac string from $\mathcal{S}$ to $\tilde{\mathcal{S}}$.

In (6.71) the parallel transports [10] are given by

$$\mathsf{S}(m,n;n') = \ U(m-1,n;\hat{1}) \prod_{q=n}^{n'} U(m,q,\hat{2})$$

and the axis of rotation

$$\vec{\theta}(m,n) = \sum_{h=1}^{\mathrm{N}-1} \theta_h \hat{n}_{[h]}(m,n) \tag{6.72}$$

[10]Here we have redefined the parallel transport $\mathsf{S}(m,n;n') \equiv \mathsf{T}^{\dagger}(m-1,n)S(m,n;n')$ and magnetic axis $\vec{n}^{\mathrm{a}}_{[1]}(m,n) \equiv R^{\mathrm{ab}}(\mathsf{T}(m-1,n))\vec{n}^{\mathrm{b}}_{[1]}(p) = \mathrm{Tr}(\Lambda^{\mathrm{a}}(U_p(m,n) + U_p^{\dagger}(m,n)))$. The advantage of using new parallel transports $\mathsf{S}(m,n;n')$ is that they are not connected to the origin and therefore more appropriate for the original Kogut-Susskind formulation.

is characterized by $\mathrm{N}-1$ external angular parameters $[\theta_1, \theta_2, ..., \theta_{\mathrm{N}-1}] = [\theta_1(m,n), \theta_2(m,n), ..., \theta_{\mathrm{N}-1}(m,n)]$. The $\mathrm{N}-1$ linearly independent magnetic axes are defined as

$$\vec{n}^{\mathrm{a}}_{[r+1]}(m,n) \equiv d^{\mathrm{abc}}\, \vec{n}^{b}_{[r]}(m,n)\, \vec{n}^{c}_{[1]}(m,n), \qquad \forall\, r = 1,2,\ldots,(\mathrm{N}-2).$$

The first magnetic axis is defined as $\vec{n}^{\mathrm{a}}_{[1]}(m,n) = \mathrm{Tr}(\Lambda^{\mathrm{a}}(U_p(m,n) + U_p^{\dagger}(m,n)))$. This operator rotate all the horizontal link $U(m-1, n'; \hat{1})$, $n' \geq n$. We can define the axis of rotation associated with each rotated link as $\vec{\Theta}^{\mathrm{a}}(m, n' > n) = R^{\mathrm{ab}}(\mathsf{S}(m,n;n'))\, \vec{\theta}^{\,\mathrm{b}}(m,n)$ which can also we recast in an iterative relation

$$\vec{\Theta}^{\mathrm{a}}(m, n'+1) = R^{\mathrm{ab}}(U(m,n';\hat{2}))\, \vec{\Theta}^{\mathrm{b}}(m,n') \tag{6.73}$$

We can similarly obtain $\Sigma^{-}_{[\vec{\theta}]}$ by using (6.4), (6.6) and (6.54). Now we have

$$\Sigma^{+}_{[\vec{\theta}]}(m,n) = \exp\left\{ i \sum_{n'=n+1}^{\infty} \vec{\Theta}^{\mathrm{a}}(m,n') \cdot E^{\mathrm{a}}_{-}(m,n';\hat{1}) \right\} \tag{6.74}$$

They rotate the links

$$\Sigma^{+}_{[\vec{\theta}]}(m,n) U_{\alpha\beta}(m,n';\hat{1}) \Sigma^{+\dagger}_{[\vec{\theta}]}(m,n) = U_{\alpha\gamma}(m,n';\hat{1}) D_{\gamma\beta}(\vec{\Theta}(m,n')), \quad \forall\, n' \geq n.$$

These rotations of the horizontal link holonomies are shown in figure 6.3-a. The rotational axes of these link holonomies are related through the parallel transport equations (6.73) which, in turn, are obtained by the exact duality transformations (6.4). These special relations ensure that they create magnetic flux only on the plaquette located at the endpoint (m,n) keeping all the other plaquette fluxes unaffected (see Appendix E). Therefore this local action by the non-local operator (6.74) creates an invisible non-Abelian Dirac string $\mathcal{S}$ originating from the corresponding plaquette (see Figure 6.3-b). In Appendix E it is shown that using gauge transformations these Dirac strings can be deformed arbitrarily except at their gauge invariant endpoints.

6.3 Path Integral Representation

In this section, we construct the path integral representation of the SU(N) disorder operators so that their behaviour can also be studied using Monte Carlo simulations in future studies. Such construction for the Z_2 't Hooft disorder operator in pure SU(2) lattice gauge theory can be found in [96, 154, 155, 158, 168, 169]. The ground state wave functional depends on the links

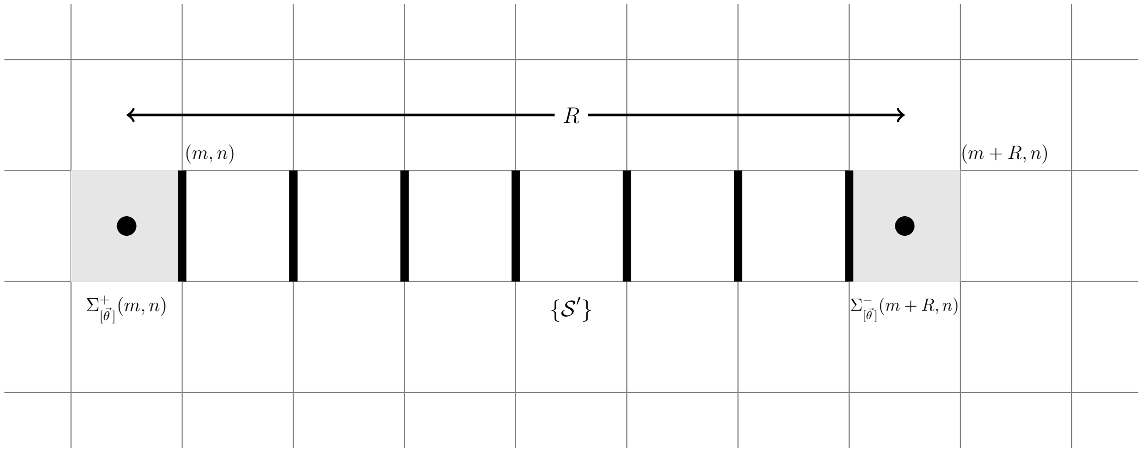


Figure 6.4: Action of the disorder operators $\Sigma^+_{[\vec\theta]}(m,n)\Sigma^-_{[\vec\theta]}(m+R,n)$ creating SU(N) vortex-antivortex at a distance R apart. The SU(N) transformations rotate the dark vertical links denoted by $\mathsf{l}' \in \mathcal{S}'$ in (6.80). This set of vertical dark links l' is denoted by $\{\mathcal{S}'\}$.

in the 2-dimensional surface Σ at time $t = 0$ [96, 168, 169]:

$$\Psi_0(U) \equiv \langle U|\psi(0)\rangle = \int \prod_{l>0} dU(l) e^{\beta \sum\limits_{p>0} \mathrm{Tr}(U_p + U_p^\dagger)} \tag{6.75}$$

In (6.75) the integration is done over all links $l > 0$ which are the links at time $t > 0$. Similarly, the plaquettes involved in the summation are in the upper half lattice at $t > 0$. Thus the ground state $\Psi_0(U)$ depends only on the spatial links at $t = 0$. The expectation values of any functional $F[U(l)]$ in the ground state $|\psi(0)\rangle$ is defined as

$$\langle F[U(l)]\rangle = \langle\psi(0)|F[U(l)]|\psi(0)\rangle.$$

The path integral representation is

$$\langle F[U(l)]\rangle = \frac{1}{Z(\beta)} \int d\mu(U) F[U(l)]\ e^{\beta \mathrm{tr}\left(U_p + U_p^\dagger\right)}, \tag{6.76}$$

where $d\mu(U) \equiv \prod_l dU(l)$ and l, p now denote all the links and plaquettes in the 3-dimensional lattice and $\beta = \frac{2\mathrm{N}}{g^2}$. The partition function $Z(\beta)$ is given by:

$$Z(\beta) = \int \prod_l dU(l) e^{\beta \sum_p (\mathrm{tr}(U_p + U_p^\dagger))} \tag{6.77}$$

The action of $\Sigma^+_{[\vec\theta]}(m,n)$ rotates all the links crossing the Dirac string by the appropriate SU(N)

Wigner D matrices as shown in Figure 6.3-a. Therefore the expectation value $\Sigma^{+}_{[\vec{\theta}]}(m,n)$ or the free energy of the SU(N) magnetic vortex can be defined as

$$\langle\, \Sigma^{+}_{[\vec{\theta}]}(m,n)\,\rangle = \left\langle e^{-\beta \sum\limits_{\mathsf{l}\in\mathcal{S}}\left[\mathrm{tr}(D(\vec{\theta})U_p + U_p^{\dagger}D^{\dagger}(\vec{\theta})) - \mathrm{tr}(U_p+U_p^{\dagger})\right]} \right\rangle \equiv \quad e^{-\beta F_{mag}(\,\vec{\theta}\,)}. \tag{6.78}$$

In (6.78) the summation sign includes only those plaquettes which protrude from the links $\mathsf{l} \in \{\mathcal{S}\}$ (see Figure 6.3-b) in the +ve time direction and $F_{mag}(\,\vec{\theta}\,)$ denotes the free energy of the magnetic vortex. Note that the path-integral representation for the SU(N) vortex (6.78) is analogous to the path integral representations for the defects in 2-d Ising model [39] and Z_{N} vortices in SU(N) gauge theory [95, 116] obtained by Kadanoff and 't Hooft respectively. We can also define SU(N) electric free energy of the vortex as the SU(N) Fourier transform

$$e^{-\beta F_{elec}(\,\vec{j}\,)} \equiv \int d\theta_1 \int d\theta_2 \cdots \int d\theta_{\mathrm{N}-1}\;\; \chi_{[\vec{j}]}(\vec{\theta})\; e^{-\beta F_{mag}(\,\vec{\theta}\,)}. \tag{6.79}$$

In (6.79), $\chi_{[\vec{j}]}(\vec{\theta})$ is the SU(N) character in the $[\vec{j}] = (j_1, j_2, \cdots, j_{\mathrm{N}-1})$ representation of SU(N).

The Monte Carlo simulation of $\langle \Sigma^{+}_{[\vec{\theta}]}(m,n)\rangle$ in (6.78) is problematic because of the presence of infinite Dirac string attached to a vortex contradicts the periodic boundary conditions imposed on a finite lattice. On the other hand one can easily compute the vortex-anti-vortex correlation functions as shown in Figure 6.4:

$$\langle \Sigma^{+}_{[\vec{\theta}]}(m,n)\; \Sigma^{-}_{[\vec{\theta}]}(m+R,n)\rangle \;\;\equiv\;\; e^{-\beta F(\vec{\theta},R)} = \;\; \left\langle e^{-\beta \sum\limits_{\mathsf{l}'\in\mathcal{S}'}\left[\mathrm{tr}(D(\vec{\theta})U_p + U_p^{\dagger}D^{\dagger}(\vec{\theta})) - \mathrm{tr}(U_p+U_p^{\dagger})\right]} \right\rangle \tag{6.80}$$

In (6.80) $\mathcal{S}'$ denotes the set of dark links l' in Figure 6.4 and the summation sign includes only those plaquettes which protrude from the links $\mathsf{l}' \in \{\mathcal{S}'\}$ in the +ve time direction. It will be interesting to study the above free energies and hence the role of SU(N) vortices in the ground state and their magnetic disorder in the large R limit using Monte Carlo simulations near the continuum $\beta \to \infty$.

CHAPTER 7

SU(N) TORIC-CODE AND NON-ABELIAN ANYONS

In this chapter, we generalize Kitaev's Z_2 toric code model to SU(N) group leading to non-Abelian anyons. We write the SU(N) analogue of Kitaev's "stabilizer operators" to write SU(N) toric-code Hamiltonian [106]. Like, Kitaev's original model, it is exactly solvable. We show that the SU(N) toric code model has N^2 degenerate ground states which are characterized by $Z_\mathrm{N} \otimes Z_\mathrm{N}$ topological charges. They are explicitly constructed in terms of SU(N) spin networks and Wigner coefficients [106]. In fact, besides topological stability on the torus, they also carry geometrical rigidity because of the triangular constraints of spin networks. All excited states are constructed using SU(N) link holonomies and generalized 't Hooft operators. We show that the SU(N) canonical commutation relations between electric fields and potentials corresponding to (non-Abelian) order-disorder algebra naturally lead to the non-Abelian anyons nature of these excitations or quasiparticles. We further show that their mutual non-Abelian statistics are encoded in Wigner D matrices. In the recent past, immense activities to realize non-Abelian lattice gauge theory Hamiltonian using cold atomic gases and optical lattices [20, 124, 170–175] will probably lead to non-Abelian toric code models as a real experimental possibility.

The organization of the chapter is as follows: In section 7.1 we construct the SU(N) toric code model and the associated operators and their algebras are discussed. In section 7.1.1 we construct the ground state in the topologically trivial sector. In section 7.1.2 we extend this construction to include $Z_\mathrm{N} \otimes Z_\mathrm{N}$ topological charges where Z_N is the centre of the group SU(N). The section 7.1.3 discusses the excited states of the SU(N) toric code Hamiltonian, excitations with non-Abelian electric charges and the non-Abelian magnetic excitations and the corresponding vortex operators [78]. In the last section, we discuss the non-Abelian anyonic

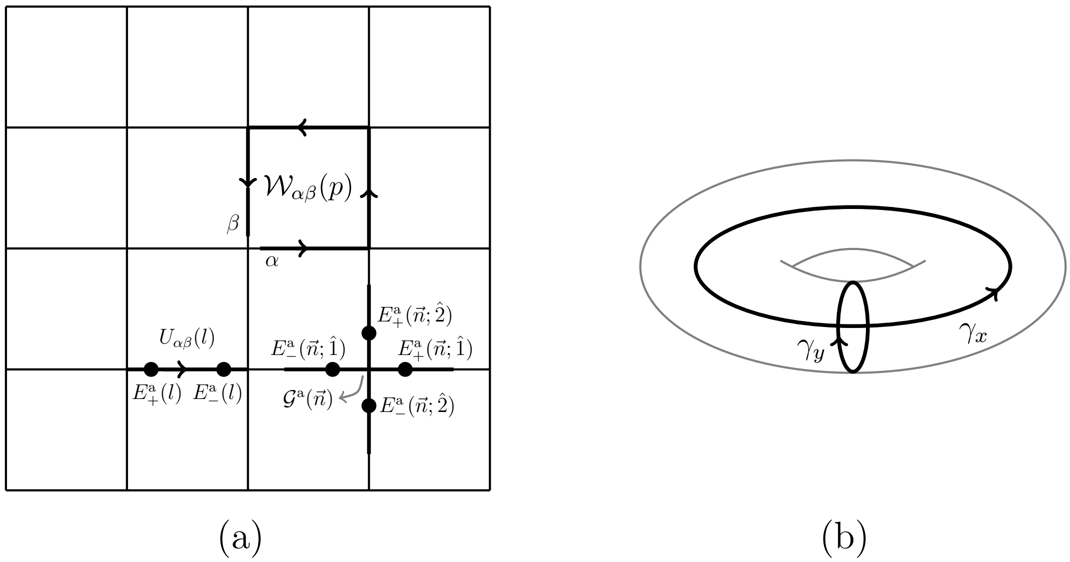


Figure 7.1: (a) The SU(N) electric fields $E^{\rm a}_{\pm}(l)$ and the conjugate link operators $U_{\alpha\beta}(l)$. The operators $\mathcal{G}^{\rm a}(\vec{n})$ and $\mathcal{W}(p)$ appearing in the Hamiltonian (7.9) are also shown, (b) Two non-contractible Wilson loops γ_x and γ_y on the torus $\mathcal{T}_2$.

nature of the above excitations.

The lattice sites on the 2-dimensional torus, denoted by $\mathcal{T}_2$, are labeled as $\vec{n} = (m, n)$; $x, y = 1, 2, \cdots, L$. The lattice links are denoted by $l \equiv (\vec{n};\, \hat{i}\,)$ with $i = 1, 2$. We will often denote the sites by v or s, links by l and plaquettes by p for shorter notations if other details are not required. On the torus $\mathcal{T}_2$ there are L^2 sites, $2L^2$ links and L^2 plaquettes. The periodicities in 2 directions imply $(x+L, y) = (x, y+L) = (m, n)$. We will often use SU(2) group for explicit calculations to avoid unnecessary group theoretical technicalities involved with SU(N), N ≥ 3.

7.1 SU(N) toric code model

The kinematical variables of SU(N) lattice theory involved in Kogut-Susskind Hamiltonian formulation [3] are SU(N) link operators $U(\vec{n}, \hat{i}) \equiv U(l)$ and the corresponding conjugate link electric fields $E^{\rm a}_{+}(\vec{n}, \hat{i}) \equiv E^{\rm a}_{+}(l)$ and $E^{\rm a}_{-}(\vec{n}+\hat{i}; \hat{i}) \equiv E^{\rm a}_{-}(l)$, see Figure 7.1-(a). These electric fields, rotate the link holonomies $U(l)$ from the left (right) end respectively leading to the following canonical commutation relations

$$[E^{\rm a}_{+}(l),\ U_{\alpha\beta}(l)] = -\,(T^{\rm a}\ U(l))_{\alpha\beta}, \qquad [E^{\rm a}_{-}(l), U_{\alpha\beta}(l)] = \ (U(l)\ T^{\rm a})_{\alpha\beta}. \tag{7.1}$$

In (7.1), $T^{\rm a}, {\rm a} = 1, 2, \cdots, {\rm N}^2 - 1$ are the generators in the fundamental representation of SU(N) satisfying ${\rm Tr}\left(T^{\rm a}T^{\rm b}\right) = \frac{1}{2}\delta^{\rm ab}$. The above commutation relations along with the Jacobi identities,

$\left[E^{\rm a}_{\pm}(l),\, [E^{\rm b}_{\pm}(l),\; U_{\alpha\beta}(l)]\right] + \left[E^{\rm b}_{\pm}(l), [U_{\alpha\beta}(l),\, E^{\rm a}_{\pm}(l)]\right] + \left[\; U_{\alpha\beta}(l),\, [E^{\rm a}_{\pm}(l),\, E^{\rm b}_{\pm}(l)]\right] = 0$ yield;

$$[E^{\rm a}_{+}(l),\; E^{\rm b}_{+}(l)] = i f^{\rm abc}\, E^{\rm c}_{+}(l), \qquad [E^{\rm a}_{-}(l),\; E^{\rm b}_{-}(l)] = \; i f^{\rm abc} E^{\rm c}_{-}(l). \tag{7.2}$$

In (7.2), $f^{\rm abc}$ are the SU(N) structure constants. The left and the right electric fields in (7.1) are related by a parallel transport [3]:

$$E^{\rm a}_{+}(l) = -R^{\rm ab}(U^{\dagger}(l))E^{\rm b}_{-}(l), \qquad \text{where} \;\; R^{\rm ab}(U) = 2{\rm tr}(T^{\rm a}U^{\dagger}T^{\rm b}U) \tag{7.3}$$

It is easy to check that on any link $l \equiv (\vec{n};\, \hat{i}\,)$, the magnitude of the left and right electric fields are equal and they commute with each other

$$\mathrm{Tr}\left(\vec{E}_{+}(l)\cdot\vec{E}_{+}(l)\right) = \mathrm{Tr}\left(\vec{E}_{-}(l)\cdot\vec{E}_{-}(l)\right) \equiv \mathrm{Tr}\; \vec{E}^{2}(l), \tag{7.4a}$$

$$\left[\vec{E}^{2}(l),\; E^{\rm a}_{+}(l)\right] = 0, \qquad \left[\vec{E}^{2}(l),\; E^{\rm a}_{-}(l)\right] = 0, \tag{7.4b}$$

$$\left[E^{\rm a}_{+}(l),\; E^{\rm b}_{-}(l)\right] = 0. \tag{7.4c}$$

The important identities (7.4a), (7.4b) and (7.4c) imply that the Hilbert space in the simplest SU(2) case is spanned by the eigenvectors $|j(l), m_{+}(l), m_{-}(l)\rangle \equiv |j, m_{+}, m_{-}\rangle_{l} = |j, m_{+}\rangle_{l} \otimes |j, m_{-}\rangle_{l}$ on every link l satisfying

$$\begin{aligned} (\vec{E}_{\pm}(l))^{2}\; |j, m_{+}, m_{-}\rangle_{l} &= j(l)\,(j(l)+1)\,|j, m_{+}, m_{-}\rangle_{l}, \\ E^{\rm a=3}_{\pm}(l)\; |j, m_{+}, m_{-}\rangle_{l} &= m_{\pm}(l)\,|j, m_{+}, m_{-}\rangle_{l}. \end{aligned} \tag{7.5}$$

The SU(N) transformations [1] on the flux operators $U(\vec{n};\, \hat{i}\,)$ and electric fields $E_{\pm}(n;\, \hat{i}\,)$ are [3]

$$U(\vec{n};\, \hat{i}\,) \rightarrow \Lambda(\vec{n})U(\vec{n};\, \hat{i}\,)\,\Lambda^{\dagger}(\vec{n}+\hat{i}\,), \quad E_{\pm}(\vec{n};\, \hat{i}\,) \rightarrow \;\; \Lambda(\vec{n})\; E_{\pm}(\vec{n};\, \hat{i}\,)\; \Lambda^{\dagger}(\vec{n}), \qquad i = 1, 2. \tag{7.6}$$

In (7.6) $\Lambda(\vec{n})$ are the SU(N) matrices describing $(\mathrm{N}^{2}-1)$ SU(N) rotational degrees of freedom at every lattice site $\vec{n}$. The canonical commutation relations (7.1) imply that the generators of the above SU(N) rotations at site $\vec{n}$ are

$$\mathcal{G}^{\rm a}(\vec{n}) \equiv \sum_{i=1}^{2}\left(E^{\rm a}_{+}(\vec{n};\, \hat{i}\,) + E^{\rm a}_{-}(\vec{n};\, \hat{i}\,)\right). \tag{7.7}$$

The Gauss law operators $\mathcal{G}^{\rm a}(n)$ are pictorially illustrated in Figure 7.1-(a). We also define SU(N) plaquette operators $\mathcal{W}(p) \equiv \mathcal{W}(\vec{n})$ which contain SU(N) magnetic fields [2] (see Figure

[1] The transformations (7.6) are the SU(N) symmetries of the toric code Hamiltonian (7.9). They do not correspond to the redundancies in SU(N) gauge theory which are removed by the Gauss law constraints $\mathcal{G}^{\rm a}(n) = 0$ at every lattice site.

[2] The SU(N) magnetic fields on plaquette p are defined as $B^{\rm a}(p) \equiv \frac{1}{2}\mathrm{Tr}[T^{\rm a}\left(2 - \mathcal{W}(p) - \mathcal{W}^{\dagger}(p)\right)]$.

7.1-a)

$$\mathcal{W}_{\alpha\beta}(p) = \left(U(\vec{n};\, \hat{1}\,)\, U(\vec{n}+\hat{1};\, \hat{2}\,)\, U^{\dagger}(\vec{n}+\hat{2};\, \hat{1}\,)\, U^{\dagger}(\vec{n};\, \hat{2}\,)\right)_{\alpha\beta} \tag{7.8}$$

We generalize Kitaev's Z_2 toric code model to SU(N) group by considering the SU(N) link flux operators $U_{\alpha\beta}(l)$ and the corresponding SU(N) electric fields $E^{\mathrm{a}}_{+}(l)$ $\left(E^{\mathrm{a}}_{-}(l)\right)$ on every link $l \equiv (\vec{n};\, \hat{i}\,)$ with $\alpha, \beta = 1, 2, \cdots, \mathrm{N}$ and $\mathrm{a} = 1, 2, \cdots, (\mathrm{N}^2 - 1)$ [3].
We now generalize Kitaev's toric code Hamiltonian and write the SU(N) toric code Hamiltonian as

$$H = A \sum_{n} A_n + B \sum_{p} B_p. \tag{7.9}$$

In the above Hamiltonian n, p denote the sites, plaquettes respectively on the torus $\mathcal{T}_2$, A, B are positive constants and

$$A_n \equiv \sum_{\mathrm{a}=1}^{\mathrm{N}^2-1} \mathcal{G}^{\mathrm{a}}(n)\mathcal{G}^{\mathrm{a}}(n), \qquad B_p \equiv 1 - \frac{1}{2\mathrm{N}} \mathrm{Tr}\left(\mathcal{W}(p) + \mathcal{W}^{\dagger}(p)\right). \tag{7.10}$$

Under SU(N) transformations (7.6) both electric and magnetic field terms in (7.7) and (7.8) respectively transform covariantly

$$\mathcal{G}(n) \to \Lambda(n)\, \mathcal{G}(n)\, \Lambda^{\dagger}(n), \qquad \mathcal{W}(p) \to \Lambda(n)\, \mathcal{W}(p)\, \Lambda^{\dagger}(n). \tag{7.11}$$

In (7.11), $\mathcal{G} \equiv \sum_{\mathrm{a}} \mathcal{G}^{\mathrm{a}} T^{\mathrm{a}}$. Therefore, toric code Hamiltonian in (7.9) is invariant under the SU(N) rotations (7.6). Like Kitaev's model, the Hamiltonian (7.9) is a sum of L^2 electric terms, L^2 magnetic terms and they all commute with each other [3]

$$[A_n, A_{n'}] = 0, \qquad [A_n, B_{p'}] = 0, \qquad [B_p, B_{p'}] = 0. \qquad \forall\ n, n', p, p'. \tag{7.12}$$

Therefore, we can diagonalize all of them simultaneously and get the exact spectrum. Before we proceed further, we note that on a torus, we have additional topological invariants defined over non-contractible loops. These non-contractible SU(N) Wilson loop operators in the fundamental representation [4] are

$$W_{\gamma_x} = \mathrm{Tr} \prod_{l\, \in \gamma_x} U(\,l\,), \qquad W_{\gamma_y} = \mathrm{Tr} \prod_{l\, \in \gamma_y} U(\,l\,) \tag{7.13}$$

In (7.13) γ_x and γ_y are the two independent oriented path shown in Figure 7.1-b and the products over links are path ordered. The paths $\gamma_x(\gamma_y)$ start from an arbitrary base point

[3]The first two relations in (7.12) are the symmetry statements of the toric code Hamiltonian (7.9). The first equation follows from the fact that $[\mathcal{G}^{\mathrm{a}}(n), \mathcal{G}^{\mathrm{a}}(n')] = 0\ \forall\, n,\, n'$ because of the identity (7.4c).

The second relation is because B_p is invariant under SU(N) rotations (7.6). The third relation is trivially true because all flux operators $U_{\alpha\beta}(l)$ commute amongst themselves.

[4]The Wilson loops in the higher representations do not lead to any new ground states.

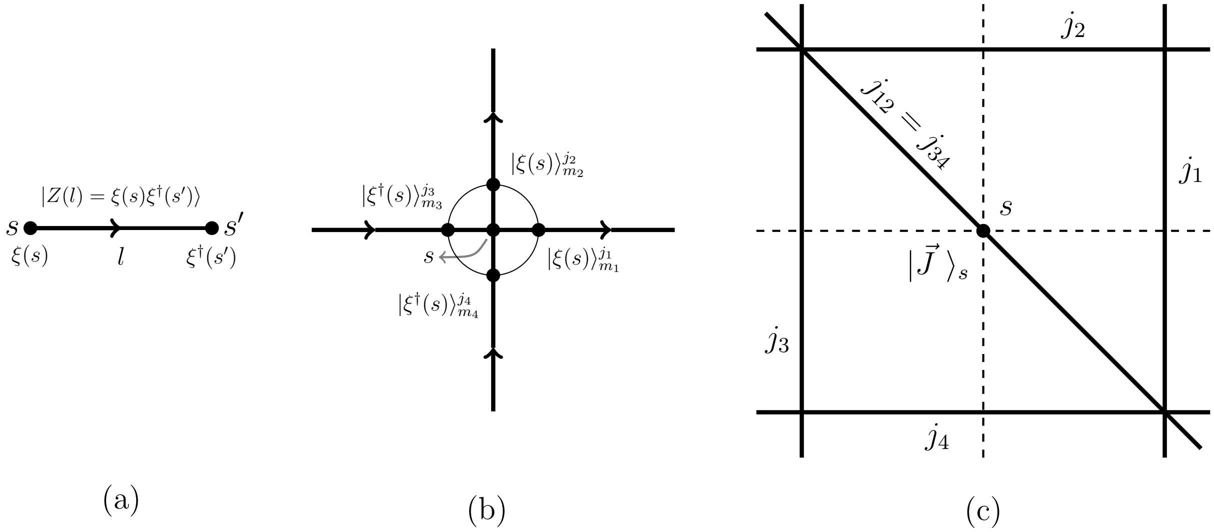


Figure 7.2: The ground states of SU(2) toric code. From (a) link states $|Z(l)\rangle$ to (b) site states $|\xi(s)\rangle^j_m$ to (c) loop states $|\vec{J}\rangle_s \equiv |j_1, j_2, j_{12} = j_{34}, j_3, j_4\rangle_s$. The geometrical triangular constraints are manifest on the dual lattice (solid lines) around the site s. Every lattice site has 2 triangular constraints: $\{j_1, j_2, j_{12}\}$ & $\{j_3, j_4, j_{34} = j_{12}\}$ must form triangles.

(x_0, y_0) and return to it after looping the torus in the horizontal (vertical) direction. These non-contractible Wilson loops can not be written as any possible product of the contractible Wilson loops operators. The shape of the non-contractible Wilson operators cab be deformed by multiplying it with contractible Wilson loops and using unitarity properties of U. Therefore we have only two distinct non-contractible loops on trous. One can introduce $Z_{\rm N}$ charges along each of these two loops which give rise to the $\rm N^2$ fold degeneracy in the ground state of the model. It is easy to check that they satisfy

$$\left[W_{\gamma_x}, W_{\gamma_y}\right] = 0,\ \left[W_{\gamma_x}, A_n\right] = 0,\ \left[W_{\gamma_x}, B_p\right] = 0,\ \left[W_{\gamma_y}, A_n\right] = \ 0,\ \left[W_{\gamma_y}, B_p\right] = 0. \tag{7.14}$$

Note that, unlike Z_2 toric code model, we do not have simple ground state projector operators for the present SU(N) generalization[5]. In the next section, we will exploit (7.12) and (7.14) to characterize the topological sectors of SU(N) toric code model and construct the $\rm N^2$ ground states with topological charges.

[5]The Z_2 toric code model has simple ground state projectors (see [107]) $\mathcal{P} \equiv \prod_n (1 + \sigma_1(l_1)\sigma_1(l_2)\sigma_1(l_3)\sigma_1(l_4)) \prod_p (1 + \sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4))$, where l_1, l_2, l_3, l_4 are the edges sharing the site n in the first term and denote the boundaries of the plaquette p in the second term. Such projectors do not exist in SU(N) toric code model.

7.1.1 Ground states

We first study the ground state structure of the SU(N) toric code Hamiltonian (7.9). A ground state $|\psi_0\rangle$ satisfies

$$A_n\,|\psi_0\rangle = 0, \qquad B_p\,|\psi_0\rangle = 0, \qquad \forall\; n, p \in \mathcal{T}_2. \tag{7.15}$$

The first condition in (7.15) enforces SU(N) rotational invariance on the ground state $|\psi_0\rangle$. These constraints are the same as (N^2-1) SU(N) Gauss law constraints in SU(N) lattice gauge theory [3, 78, 79]. Therefore, all possible ground states of the SU(N) toric code Hamiltonian are the all mutually independent loop states of the SU(N) lattice gauge theory with $B_p = 0$. These loop states are best analyzed and characterized in the dual electric flux basis [78]. We note that in two space dimensions, there are 4 SU(N) electric fluxes meeting at a vertex. Their quantum numbers can be characterized by the corresponding 4 SU(N) Young tableaues. The Gauss law constraints $A_v = 0$ states that the ground state manifold consists of all possible independent SU(N) invariant states with total SU(N) electric flux = 0. In the simple N = 2 case [67, 71–73, 137], a basis in the loop Hilbert space at a vertex s is given by $|j_1, j_2, j_{12} = j_{34}, j_3, j_4\rangle_s \equiv |\vec{J}\rangle_s$. This is shown in Figure 7.2-c. They satisfy

$$\mathcal{G}^{\mathrm{a}}(s)\;|\vec{J}\rangle_s = 0, \qquad \forall s \in \mathcal{T}_2. \tag{7.16}$$

Therefore, if we label the L^2 vertices or sites on the torus by $v_1, v_2, \cdots, v_{L^2}$ then the ground state of SU(2) toric code is of the form:

$$|\psi_0\rangle = \sum_{\{\vec{J}_{v_1},\vec{J}_{v_2},\cdots,\}} \underbrace{W\left(\vec{J}_{v_1}, \vec{J}_{v_2}, \cdots,\right)}_{\text{amplitude on torus}} \underbrace{\left\{\prod_{i=1}^{L^2} \otimes\,|\vec{J}\rangle_{v_i}\right\}}_{\text{loop states on torus}} \tag{7.17}$$

In (7.17) the summation over $\{\vec{J}_{v_i}\}$ are constrained as each electric flux $j(l)$ is shared by two vertices at the 2 ends of the link l. The first ground state condition $A_v|\psi_0\rangle = 0$ is ensured by the spin network identities (7.6) for arbitrary amplitudes $W\left(\vec{J}_{v_1}, \vec{J}_{v_2}, \cdots,\right)$. These amplitudes are now fixed by demanding $B_p|\psi_0\rangle = 0$. However, we find this approach difficult as the action of B_p on a spin network state is not simple [67,71–73,137]. We, therefore, reverse the process and start with completely ordered magnetic eigenstates with $B_p = 0,\ \forall\, p$ and then ensure $A_v|\psi_0\rangle = 0$ by demanding invariance under the symmetry transformations (7.6) at every vertex. We first notice that the eigenstates of B_p in (7.10) are necessarily eigenstates of the individual link holonomies $U(l)_{\alpha\beta}$. As all these flux operators commute with each other $[U_{\alpha\beta}(l), U_{\gamma\delta}(l')] = 0,\ \forall\ l, l'$ and $\forall\ \alpha, \beta, \gamma, \delta$, we can diagonalize all of them simultaneously. Further the eigenvalues of any SU(2) matrices operator $U_{\alpha\beta}(l)$ are SU(2) matrices therefore we define SU(2) group manifold S^3 on every link l:

$$Z(l) = \begin{bmatrix} z_1(l) & z_2(l) \\ -z_2^*(l) & z_1^*(l) \end{bmatrix}. \tag{7.18}$$

In (7.18) $(z_1, z_2) \in S^3$ and satisfy $|z_1(l)|^2 + |z_2(l)|^2 = 1,\ Z(-l) \equiv Z^\dagger(l), \forall\ l$. We define the magnetic eigenstates to be eigenstates of each link operator:

$$U_{\alpha\beta}(l)\ |Z(l)\rangle = Z_{\alpha\beta}(l)\ |Z(l)\rangle. \tag{7.19}$$

The eigenvectors are

$$|Z(l)\rangle \equiv |z_1(l), z_2(l)\rangle = \frac{1}{4\pi}\sum_{j=0}^{\infty}\sqrt{(2j+1)}\sum_{m_\pm} D^{\ j}_{m_+,m_-}(Z(l))\ |j\ m_+\rangle_l \otimes |j\ m_-\rangle_l. \tag{7.20}$$

To obtain ground state with $B_p = 0$ or equivalently $\mathcal{W}_{\alpha\beta}(p) = \delta_{\alpha\beta}$ we choose pure gauge conditions on every link to write

$$Z(l) \equiv Z(\vec{n}, \hat{i}) = \xi(s)\ \xi^\dagger(s') \tag{7.21}$$

as shown in Figure 7.2-a. At each site $s \in \mathcal{T}_2$, $(\xi(s) \equiv \xi_1(s), \xi_2(s)) \in S^3$ with $|\xi_1(s)|^2 + |\xi_2(s)|^2 = 1$ and can be written as SU(2) matrix (7.18). We can thus write the maximally ordered ($\mathcal{W}_p = 1,\ \forall\, p$) magnetic eigenstates in (7.20) as the direct product of site states defined at the two end sites of the link l,

$$|Z(l)\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} |\xi(s)\rangle^j_m \otimes |\xi^\dagger(s')\rangle^j_m. \tag{7.22}$$

The site states at left and right ends (s and s') of the link l are defines as

$$\begin{aligned} |\xi(s)\rangle^j_m &= \sqrt{d_j}\sum_{m_+} D^j_{m_+,m}(\xi(s))\ |j,\, m_+\rangle_l, \\ |\xi^\dagger(s')\rangle^j_m &= \sqrt{d_j}\sum_{m_-} D^j_{m,m_-}(\xi^\dagger(s'))\ |j,\, m_-\rangle_l \end{aligned} \tag{7.23}$$

Above $d_j \equiv \sqrt{(2j+1)}/4\pi$. Therefore, all states satisfying $B_p|\psi\rangle = |\psi\rangle$ can be written as product of the site states at every site s

$$|\psi\rangle = \sum_{\text{all } j}\ \sum_{\text{all } m}\ \prod_{s\in\mathcal{T}_2} |\xi(s)\rangle^{j_1}_{m_1} \otimes |\xi(s)\rangle^{j_2}_{m_2} \otimes |\xi^\dagger(s)\rangle^{j_3}_{m_3} \otimes |\xi^\dagger(s)\rangle^{j_4}_{m_4}, \tag{7.24}$$

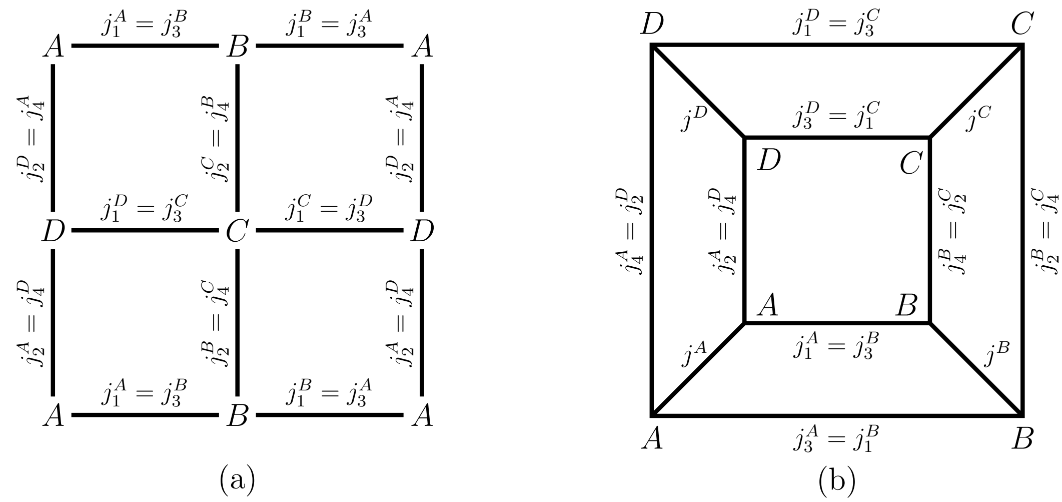


Figure 7.3: SU(2) toric code. (a) Electric fluxes on a 4 plaquette toric code, (b) 12-j Wigner coefficients involving 8 triangular constraints as amplitude for SU(2) toric code ground state.

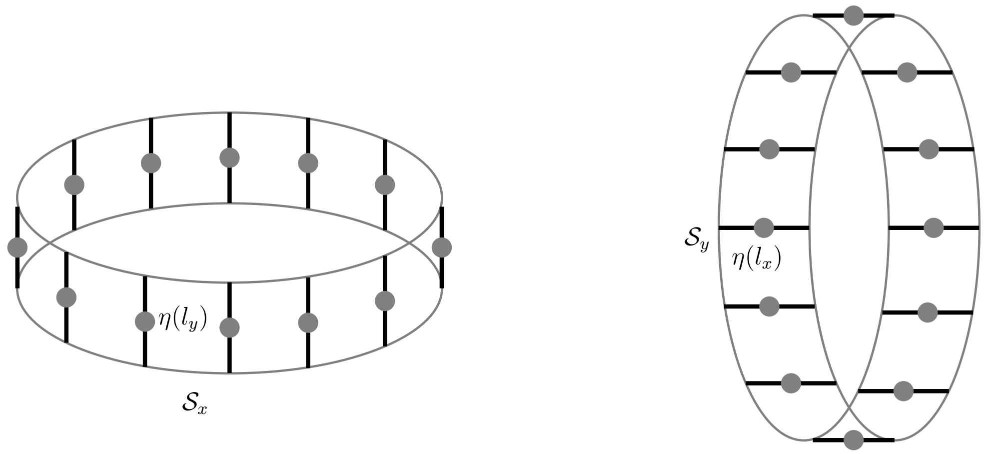
We can now perform the integrations over $\xi(s)$ at every site s to get the loop states satisfying both the conditions in (7.15). Using the multiplicative properties of the Wigner rotation matrices [176], these group manifold integrations at every lattice site can be performed exactly to get

$$
\begin{aligned}
|\psi_0\rangle &= \sum_{\text{all } j}\sum_{\text{all } m}\prod_{s\in\mathcal{T}_2}\left\{\int_{S^3} d^2\mu(\xi(s))\ |\xi(s)\rangle^{j_1}_{m_1}\otimes|\xi(s)\rangle^{j_2}_{m_2}\otimes|\xi^\dagger(s)\rangle^{j_3}_{m_3}\otimes|\xi^\dagger(s)\rangle^{j_4}_{m_4}\right\}\\
&= \sum_{\{\vec{J}\}} W\{\vec{J}\} \quad \prod_s \otimes \underbrace{|j_1, j_2, j_{12}=j_{34}, j_3, j_4\rangle_s}_{\text{loop state at site s}}\\
&= \sum_{\{\vec{J}\}} W\{\vec{J}\} \quad \prod_s \otimes\ |\vec{J}\rangle_s. \qquad (7.25)
\end{aligned}
$$

In (7.25) the loop states at every site s are

$$
|\vec{J}\rangle_s = \sum_{all\ m's} \eta^{j_{12}}_{m_{12}} C^{j_{12}m_{12}}_{j_1m_1,j_2m_2} C^{j_{12},-m_{12}}_{j_3m_3,j_4m_4} \prod_{i=1}^{4}\otimes|j_im_i\rangle \qquad (7.26)
$$

In (7.26) $\eta^j_m \equiv (-1)^{j-m}(2j+1)^{-\frac{1}{2}}$ and the loop states $|\vec{J}\rangle_s \equiv |j_1, j_2, j_{12} = j_{34}, j_3, j_4\rangle_s$ contain two group theoretical triangular constraints [137] around every lattice site. They are illustrated on the dual lattice plaquette in Figure 7.2-c. The amplitudes $W\{\vec{J}\}$ can be computed after integrating over all L^2 lattice sites and then summing over the remaining magnetic quantum numbers. We note that all the magnetic quantum numbers are completely contained in the



(a) (b)

Figure 7.4: Construction of topologically nontrivial ground states. Inserting Z_{N} twist factors, shown as ●, in the ground states: (a) Z_{N} phase η_y along $\mathcal{S}_x$, (b) Z_{N} phase η_x along $\mathcal{S}_y$.

Clebsch-Gordan coefficients appearing as the coefficients of the loop states

$$\int_{S^3} d^2\mu(\xi(s))\, |\xi(s)\rangle^{j_1}_{m_1} \otimes |\xi(s)\rangle^{j_2}_{m_2} \otimes |\xi^\dagger(s)\rangle^{j_3}_{m_3} \otimes |\xi^\dagger(s)\rangle^{j_4}_{m_4}$$
$$= K_{\{j_1,j_2,j_3,j_4\}}(-1)^{j_3-m_3+j_4-m_4} \sum_{j_{12},m_{12}} \eta^{j_{12}}_{m_{12}} C^{j_{12},m_{12}}_{j_1,m_1\, j_2,m_2} \times C^{\,j_{12},-m_{12}}_{j_3,m_3\, j_4,m_4} |j_1, j_2, j_{12}, j_3, j_4\rangle_s,$$

where $K_{\{j_1,j_2,j_3,j_4\}} = \prod_{\{\text{all } j\}} (2j+1)^{1/4}$. The sums over remaining magnetic quantum numbers (m_1, m_2, m_3, m_4) attached to every site can be done after integration over all L^2 lattice sites on $\mathcal{T}_2$. These amplitudes ensure that the triangular flux constraints are satisfied at every lattice site and they are also glued together on the entire torus $\mathcal{T}_2$. It is illustrative to consider a simple example and understand this construction. We consider a torus with 4 lattice sites A, B, C and D and 4 associated plaquettes as shown in Figure 7.3-a. The ground state in (7.25) can now be written as

$$|\psi_0\rangle = \sum_{\{\vec{J}_A,\vec{J}_B,\vec{J}_C,\vec{J}_D\}} \underbrace{W\left(\vec{J}_A, \vec{J}_B, \vec{J}_C, \vec{J}_D\right)}_{\text{amplitude}} \underbrace{|\vec{J}_A\rangle \otimes |\vec{J}_B\rangle \otimes |\vec{J}_C\rangle \otimes |\vec{J}_D\rangle}_{\text{loop states on torus}}. \tag{7.27}$$

The amplitudes $W(\vec{J}_A, \vec{J}_B, \vec{J}_C, \vec{J}_D)$ in (7.27) are the 12-j Wigner coefficients of the 2nd kind:

$$W\left(\vec{J}_A, \vec{J}_B, \vec{J}_C, \vec{J}_D\right) = \sqcap \begin{bmatrix} j^D_4 = j^A_2 & & j^A_1 = j^B_3 & & j^B_4 = j^C_2 & & j^C_1 = j^D_3 & \\ & j^A & & j^B & & j^C & & j^D \\ j^D_2 = j^A_4 & & j^A_3 = j^B_1 & & j^B_2 = j^C_4 & & j^C_3 = j^D_1 & \end{bmatrix} \tag{7.28}$$

We have defined $\Pi = \prod_{(\text{all } j)} \sqrt{(2j+1)}$. The $12j$ Wigner coefficients are shown in Figure 7.3-b. We this have $A_v|\psi_0\rangle = 0$ because of the spin network structure in (7.26) and $B_p|\psi_0\rangle = 0$ because of the $12j$ coefficients in (7.28). The pure gauge conditions (7.21) make the ground state $|\psi_0\rangle$ satisfy

$$W_{\gamma_x}|\psi_0\rangle = |\psi_0\rangle, \qquad W_{\gamma_y}|\psi_0\rangle = |\psi_0\rangle. \tag{7.29}$$

Hence $|\psi_0\rangle$ is in the trivial topological sector. We now construct states with topological charges.

7.1.2 $Z_{\mathbf{N}} \otimes Z_{\mathbf{N}}$ charges

To construct topologically nontrivial ground states carrying $Z_{\mathrm{N}} \otimes Z_{\mathrm{N}}$ charges, we generalize the pure gauge conditions (7.21) as follows. We attach $Z_{\mathrm{N}} \otimes Z_{\mathrm{N}}$ phase factors η_x and η_y to all the links along the two strips $\mathcal{S}_y$ and $\mathcal{S}_x$ encircling the torus as shown in Figure 7.4 -a,b.

$$\begin{aligned} Z(l_x) &= \xi(s)\,\eta(l_x)\,\xi^{\dagger}(s'), \quad \eta(l_x) \equiv \eta_x = e^{\frac{2\pi i \mathsf{p}}{\mathrm{N}}}, \quad l_x \in \mathcal{S}_y, \\ Z(l_y) &= \xi(s)\,\eta(l_y)\,\xi^{\dagger}(s'), \quad \eta(l_y) \equiv \eta_y = e^{\frac{2\pi i \mathsf{q}}{\mathrm{N}}}, \quad l_y \in \mathcal{S}_x. \end{aligned} \tag{7.30}$$

In (7.30), $\mathsf{p}, \mathsf{q} = 0, 1, 2, \cdots, (\mathrm{N}-1)$. Repeating the procedure of the last section after replacing (7.21) with (7.30) we get

$$|\psi_0\rangle_{(\mathsf{p},\mathsf{q})} = \sum_{\{\vec{J}\}} (\eta_x)^{\{\mathcal{J}_x\}} \, (\eta_y)^{\{\mathcal{J}_y\}} \, W\{\vec{J}\} \prod_s \otimes \underbrace{|j_1, j_2, j_{12} = j_{34}, j_3, j_4\rangle_s}_{\text{loop state at site s}}. \tag{7.31}$$

The additional phase factors in (7.31) leading to topologically non-trivial sectors are

$$\mathcal{J}_x = \sum_{l_x \in \mathcal{S}_y} j(l_x), \qquad \mathcal{J}_y = \sum_{l_y \in \mathcal{S}_x} j(l_y). \tag{7.32}$$

The ground states $|\psi_0\rangle_{(\mathsf{p},\mathsf{q})}$ satisfy

$$W_{\gamma_x}|\psi_0\rangle_{(\mathsf{p},\mathsf{q})} = \eta_x|\psi_0\rangle_{(\mathsf{p},\mathsf{q})}, \qquad W_{\gamma_y}|\psi_0\rangle_{(\mathsf{p},\mathsf{q})} = \eta_y|\psi_0\rangle_{(\mathsf{p},\mathsf{q})}. \tag{7.33}$$

The ground state in the trivial sector in (7.25) is $|\psi_0\rangle = |\psi\rangle_{(\mathsf{p}=0,\,\mathsf{q}=0)}$. Having constructed all N^2 ground states, we now discuss the quasiparticle excited states and then their non-Abelian anyonic nature.

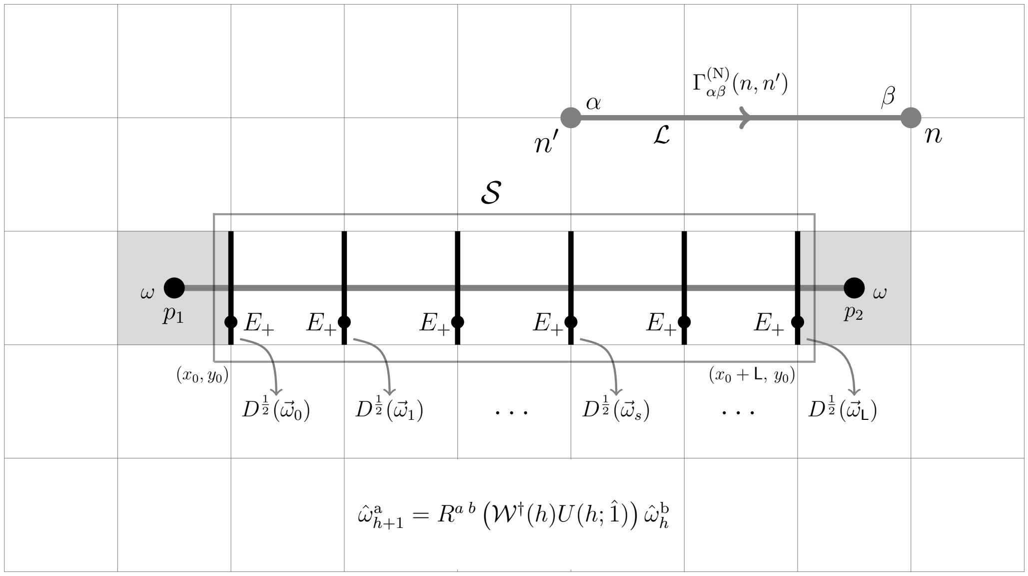


Figure 7.5: Creating electric charges at sites and magnetic vortices on plaquettes. The operator $\Gamma^{(N)}_{\alpha\beta}(n', n)$ creates electric charges in the fundamental representations at the two end points n and n'. The operator $\Sigma_{\vec{\omega}}$ acts on all the vertical link fluxes $U(s; \hat{2}) \in \mathcal{S}$ (dark vertical links) and rotates all of them by the same angle $\omega(x_0, y_0) = \omega_{s=0} \equiv \omega$. However, the rotation axes $\hat{\omega}_{s=1}, \hat{\omega}_{s=2}, \cdots, \hat{\omega}_{s=L}$ are different and related to $\hat{\omega}_0$ by parallel transports (7.41). Note that the shapes and lengths of $\mathcal{L}$ and $\mathcal{S}$ are unphysical.

7.1.3 Non-Abelian electric and magnetic excitations

In this section, we construct all possible excitations carrying SU(N) electric and magnetic fluxes. In the next section, we will construct the SU(N) anyonic states in this toric code model.

[A] Electric Fluxes

We now define the Wilson charge operator

$$\Gamma^{(N)}_{\alpha\beta}(n', n) = \left(\prod\nolimits^{p}_{l\in\mathcal{L}} U(l)\right)_{\alpha\beta}. \tag{7.34}$$

In (7.34) $\prod^{p}$ denotes the lattice path ordered product of link holonomies in the SU(N) fundamental representation along the oriented path $\mathcal{L}$ as shown in figure 7.5. The string operator is invariant under rotations (7.6) all along $\mathcal{L}$ except at the end point n (n') where it creates SU(N) quasiparticle in the fundamental N (anti-fundamental $\bar{\text{N}}$) representation.

$$\left[A_v, \Gamma^{(N)}_{\alpha\beta}(n', n)\right] = \begin{cases} \frac{(N^2-1)}{N^2}\ \Gamma^{(j)}_{\alpha\beta}(n', n), & \text{if } \ v = n \ or \ n', \\ 0 & \text{otherwise} \end{cases} \tag{7.35}$$

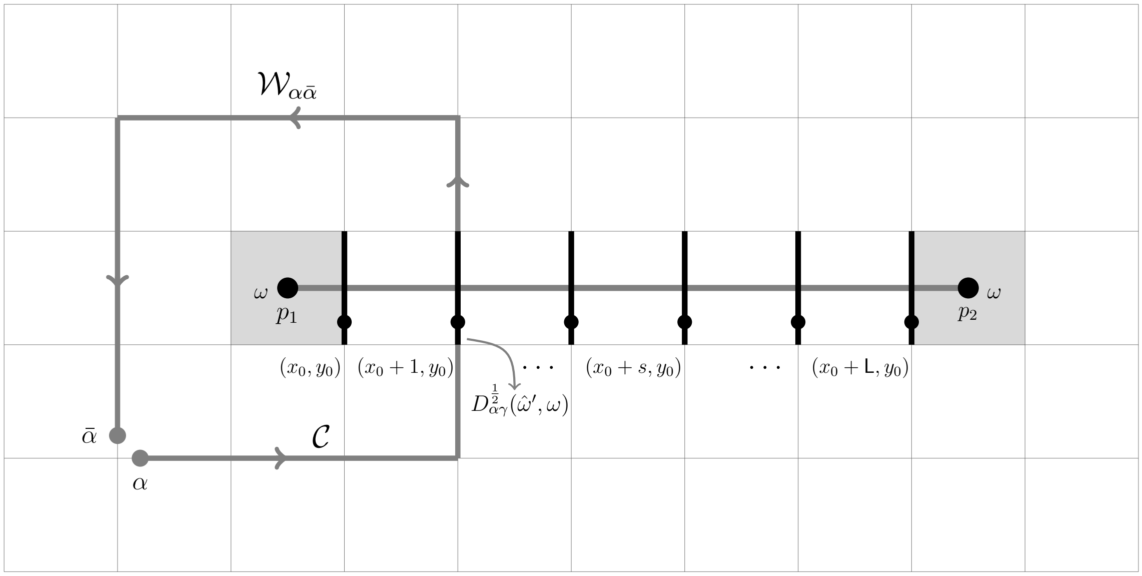


Figure 7.6: Taking a charge around close loop $\mathcal{C}$ produces a non-Abelian rotation by Wigner rotation matrix if a vortex is enclosed as shown in (7.48).

We also note that the length and shape of the string $\mathcal{L}$ between the endpoints are invisible and therefore unphysical. As $U_{\alpha\beta}(l)$ commute amongst themselves,

$$\Big[B_p,\ \Gamma^{(\mathrm{N})}_{\alpha\beta}(n', n)\Big] = 0. \qquad \forall\ p;\ n', n. \tag{7.36}$$

Thus the quasiparticle states

$$|\psi^{(\mathrm{N})}_{\alpha\beta}(n', n)\rangle_{(\mathsf{p},\mathsf{q})} \equiv \Gamma^{(\mathrm{N})}_{\alpha\beta}(n', n)\ |\psi_0\rangle_{(\mathsf{p},\mathsf{q})} \tag{7.37}$$

are the eigenstates of the SU(N) toric code Hamiltonian (7.9)

$$H\,|\psi^{(\mathrm{N})}_{\alpha\beta}(n', n)\rangle_{(\mathsf{p},\mathsf{q})} = 2A\,\frac{(\mathrm{N}^2-1)}{\mathrm{N}^2}\,|\psi^{(\mathrm{N})}_{\alpha\beta}(n', n)\rangle_{(\mathsf{p},\mathsf{q})}. \tag{7.38}$$

We have obtained the above eigenvalue equation using (7.35). Therefore, $|\psi^{(\mathrm{N})}_{\alpha\beta}(n', n)\rangle_{(\mathsf{p},\mathsf{q})}$ are the eigenstates of H representing quasiparticles in the fundamental (anti-fundamental) representations at lattice sites n (n'). We can similarly construct electric states in the higher representations of SU(N). These state will be characterized by the corresponding SU(N) Young tableau. Only ends points of the electric and magnetic excitation are observable. Thus the two ends points of the strings behave like two independent particles.

[B] Magnetic fluxes

We now construct unitary vortex operators which create and destroy non-Abelian magnetic fluxes on two plaquettes [78]. The SU(2) magnetic fluxes on the plaquettes in the axis, angle representation $\vec{\omega} \equiv (\hat{\omega}, \omega)$ can be written as

$$\mathcal{W} = \cos\left(\frac{\omega}{2}\right)\sigma_0 + i\,\hat{\omega}\cdot\vec{\sigma}\ \sin\left(\frac{\omega}{2}\right). \tag{7.39}$$

To construct a vortex or disorder operator creating the above magnetic flux we consider a finite-length ladder strip $\mathcal{S}$ which extends from the plaquette p_1 to the plaquette p_2 as shown in Figure 7.5. These vortex operators are the generalization of 't Hooft vortex operators [1] which create Z_{N} center flux vortices on a plaquette. We will see that like the invisible string $\mathcal{L}$ associated with the Wilson charge operators (7.34), the path $\mathcal{S}$ is also invisible and only the locations of the end points (plaquettes p_1, p_2) matter. We now write the SU(2) vortex pair creation operator as

$$\Sigma_{\vec{\omega}}(p_2, p_1) = \exp i\left(\sum_{s=0}^{\mathsf{L}} \hat{\omega}_s \cdot \vec{E}_+(s;\,\hat{2})\right)\omega. \tag{7.40}$$

In the above equation $E_+^{\mathrm{a}}(s;\,\hat{2})$ are the left or bottom electric fields on the vertical link $U(s;\hat{2}) \equiv U(x_0+s, y_0;\,\hat{2}) \in \mathcal{S}$. The axes of rotations $\hat{\omega}_s$ along the ladder strip $\mathcal{S}$ are related to $\hat{\omega}_{s=0} \equiv \hat{\omega}(x_0, y_0) = \hat{\omega}$ by parallel transports

$$\hat{\omega}^{\mathrm{a}}_{h+1} = R^{\mathrm{ab}}\left(\mathcal{W}^{\dagger}(h)U(h;\hat{1})\right)\hat{\omega}^{\mathrm{b}}_{h},\ \ h = 0, 1, \cdots, \mathsf{L}-1 \tag{7.41}$$

We have used notation; $U(h;\,\hat{1}) \equiv U(x_0+h, y_0;\hat{1})$ and $\mathcal{W}(h) \equiv \mathcal{W}(x_0+h, y_0)$ where the plaquette operators $\mathcal{W}(m,n)$ are defined in (7.8). The vortex operator $\Sigma_{\vec{\omega}}(p_2, p_1)$ creates magnetic vortices on the plaquettes p_1 and p_2 which are located at the end points of the ladder strip $\mathcal{S}$. All other plaquette magnetic fluxes along $\mathcal{S}$ remain zero [78]. Therefore, we can define magnetic vortex states as

$$|\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})} \equiv \Sigma_{\vec{\omega}}(p_2, p_1)\ |\psi_0\rangle_{(\mathsf{p},\mathsf{q})}. \tag{7.42}$$

These are the states with magnetic vortices on the plaquettes p_1 and p_2 as they satisfy (see Appendix E)

$$B_p\ |\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})} = \begin{cases} \left(1-\cos\left(\frac{\omega}{2}\right)\right)\ |\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})}, & \text{if } p = p_1 \text{ or } p_2 \\ |\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})}, & \text{otherwise.} \end{cases} \tag{7.43}$$

The vortex operator $\Sigma_{\vec{\omega}}(p_2, p_1)$ is invariant under (7.6)

$$A_n \, |\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})} = 0, \qquad \forall\, n,\; p_1, p_2. \tag{7.44}$$

Therefore, they are the eigenstates of the toric code Hamiltonian (7.9)

$$H \, |\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})} = 4B \left(1 - \cos\left(\frac{\omega}{2}\right)\right) \, |\psi_{\vec{\omega}}(p_2, p_1)\rangle_{(\mathsf{p},\mathsf{q})}.$$

We note that, unlike the electric charge sector, the magnetic vortex spectrum is continuous. However, a gap can be created in the magnetic sector also by adding an electric-magnetic interaction term in (7.9) which involves both A_n and B_p. For example, using $1-1$ correspondence between lattice sites (n) and plaquettes (p) on $\mathcal{T}_2$, we add the following term in the Hamiltonian

$$\Delta H = C \sum_n \left(\alpha A_n - B_p\right)^2, \tag{7.45}$$

where C and α are positive constants. The above interaction term leaves the N^2 ground states and the exact solvability of the model intact. However, in the large C limit, the energetically allowed low energy excitations become discrete and their energies can be manipulated by tuning the parameter α.

7.1.4 SU(N) Anyonic states

After having discussed the N^2 degenerate ground states and their electric, magnetic excitations, we now briefly analyze their non-Abelian anyonic nature. As we will see, the non-Abelian nature originates from the basic non-Abelian canonical commutation relations (7.1). This is also the case with the Kitaev's Z_2 toric code model. We consider taking an electric charged particle in the fundamental representation around a loop $\mathcal{C}$ in the presence of magnetic vortices at p_1 and p_2. This is shown in Figure 7.6. Using the canonical commutation relations between the electric fields and the potentials (7.1) we get

$$\Sigma_{\vec{\omega}}(p_2, p_1)\mathcal{W}_{\alpha\beta}(\mathcal{C})\; \Sigma_{\vec{\omega}}^{-1}(p_2, p_1) \;=\; \begin{cases} D^{j=\frac{1}{2}}_{\alpha\gamma}\,(\hat{\omega}', \omega)\;\; \mathcal{W}_{\gamma\beta}(\mathcal{C}), & \text{if } \mathcal{C} \;\text{ encloses } p_1, \\ D^{j=\frac{1}{2}}_{\alpha\gamma}\,(\hat{\omega}'', \omega)\;\; \mathcal{W}_{\gamma\beta}(\mathcal{C}), & \text{if } \mathcal{C} \;\text{ encloses } p_2, \\ \mathcal{W}_{\alpha\beta}(\mathcal{C}) \text{ otherwise.} & \end{cases} \tag{7.46}$$

The algebra (7.46) is the generalized Wilson-'t Hooft order-disorder algebra [78, 95] (see Appendix E for details). Thus an electric charge encircling a vortex undergoes SU(2) rotation by Wigner matrix [6]. This can be seen as follows. We consider the following initial state $|I\rangle$ and

[6] In (7.46) the unit vector $\hat{\omega}'$ and $\hat{\omega}''$ are related to unit vector $\hat{\omega}$ through parallel transports which depend on the shapes, sizes of the string $\mathcal{S}$ as well as $\mathcal{C}$. Therefore, axes orientations are unphysical. However, the

the final state $|F\rangle$:

$$
\begin{aligned}
|I\rangle_{\alpha\beta}^{(\omega)} &\equiv \Sigma_{\vec{\omega}}\ \Gamma_{\alpha\beta}^{(j=1/2)}(n,n')\ |\psi_0\rangle_{\mathsf{p},\mathsf{q}}, \\
|F\rangle_{\alpha\beta}^{(\omega)} &\equiv \mathcal{W}_{\alpha\bar{\alpha}}(\mathcal{C})\ \Sigma_{\vec{\omega}}\ \Gamma_{\bar{\alpha}\beta}^{(j=1/2)}(n,n')\ |\psi_0\rangle_{\mathsf{p},\mathsf{q}}.
\end{aligned}
\tag{7.47}
$$

Both the initial and the final states are in the (p,q) sector. Using (7.46) we get

$$
|F\rangle_{\alpha\beta}^{(\omega)} = \begin{cases} D_{\alpha\bar{\alpha}}^{j=\frac{1}{2}}(\hat{\omega}',\omega)\,|I\rangle_{\bar{\alpha}\beta}^{(\omega)}, & \text{if } \mathcal{C} \text{ encloses a vortex,} \\ |I\rangle_{\alpha\beta}^{(\omega)}, & \text{otherwise.} \end{cases}
\tag{7.48}
$$

The above result is illustrated in Figure 7.6. We note that the topological charges (p,q) are not detected by the Wilson loop $\mathcal{W}(\mathcal{C})$ in (7.48). This is an expected result as no local measurement can distinguish different topological sectors. We can also consider the electric charges in the higher j representations leading to the Wigner rotation matrices $D_{mm'}^{j}(\hat{\omega}',\omega)$ as the matrix on the right-hand side of (7.46). We note that the composites of Γ and Σ have both electric and magnetic charges and behave like dyons. Their mutual interchange or braiding will lead to non-Abelian anyonic statistics.

magnitude ω is invariant under (7.6) and is also robust against any deformations or choices of the closed curve $\mathcal{C}$ or $\mathcal{S}$ in Figure 7.6.

CHAPTER 8

DISCUSSION & CONCLUSIONS

In the past there have been various ansatze for solving the SU(N) Gauss law constraints for N = 2 [67, 70, 146], N = 3 [145] and write the theory in terms of the dual variables [48, 67, 69]. There are also different ansatze to construct SU(2) gauge invariant (loop) basis [71–77]. However, our ideas and approach discussed in this thesis are completely different from the above approaches [78, 79, 91]. We use canonical transformations over the entire two-dimensional spatial lattice to obtain exact duality. We have shown that this simple approach works for spin models (Kramers-Wannier duality), spin gauge theory (Wegner duality) and Abelian as well as non-Abelian SU(N) lattice gauge theories. The dual operators emerge naturally through canonical transformations. Note that no ansatz was used to obtain the exact solutions of Z_2 or SU(N) Gauss law constraints (see (2.33) and (4.8) respectively). We also show that these duality ideas may also lead to their practical applications in the study of phase transitions and topological quantum computing.

The other main result discussed in the thesis is the taming of the non-locality in the dual formulation of SU(N) lattice gauge theory. The origin of this nonlocality is that the non-Abelian plaquette variables are not gauge-invariant and transform as SU(N) adjoint matter fields (see 5.10). Therefore they require (auxiliary) gauge fields to interact locally through minimal couplings. This was done in the chapter (5) where the missing horizontal gauge fields were introduced using the new plaquette constraints (5.2.1). These constraints are crucial as they restore the manifest locality and the manifest rotational invariance of the original Kogut-Susskind formulation (3.1) after duality (5.2). The final SU(N) duality transformations (5.36) are expected covariant generalizations of the corresponding Abelian duality transformations (2.69). Note that the above nonlocality or rotational invariance complications do not arise in the simple Abelian case as the Z_2 or U(1) plaquettes are gauge invariant and do not need gauge fields for mutual interactions. The resulting local dual dynamics is described by the local Hamiltonian (5.41).

The canonical transformations convert the plaquette interaction terms $\frac{1}{g^2}\mathrm{Tr}(U_1U_2U_3^\dagger U_4^\dagger + h.c.)$ into pure magnetic field terms $\frac{1}{g^2}\mathrm{Tr}(\mathcal{W}(m,n) + h.c.)$ and the pure electric field terms get dualized into electric scalar potential minimal coupling interaction terms. These results are important as the plaquette interaction terms involving 4 links, dominating near the continuum $g^2 \rightarrow 0$ limit, have been completely simplified. In the past, even in the simple SU(2) lattice gauge theory case, these plaquette interactions become extremely complicated in the gauge invariant loop Hilbert space [29,38,48,49,61,62,67,69–77,137]. Therefore, it will be interesting to develop a systematic weak coupling loop perturbation theory in the $g^2 \rightarrow 0$ continuum limit with the dual Hamiltonian (5.41).

In the $(3+1)$ dimension these canonical transformations can be carried out on every (XZ) and (YZ) plane similar to the present $(2+1)$ dimensional case. We thus convert all X, Y-links at $z > 0$ into (XZ), (YZ) plaquettes respectively and Z-links into the unphysical strings. Now the dual formulation will have nonlocality in both the electric and magnetic field parts of the Hamiltonian. The nonlocality in the electric field part, like in $(2+1)$ dimension, will be due to the gauge invariance, whereas the absence of (XY)-plaquette will introduce nonlocality in the magnetic part. These non-local dynamics can again be made local by introducing new plaquette constraints. At present this work is in progress.

APPENDIX A

MANDELSTAM CONSTRAINTS

A gauge theory's physical observables and physical states should be gauge invariant and satisfy Gauss law constraints. Therefore, it is useful to remove the gauge redundancies by reformulating the theory in terms of a set of gauge invariant Wilson loops, $W(\Gamma) = \mathrm{Tr}\left(\mathrm{P}e^{i\int_{\Gamma} A_\mu dx^\mu}\right)$, where Ai is the gauge connection and Γ is an oriented, closed loop. It was shown in [57] that the set of all Wilson loops contains all the gauge invariant information contained in a gauge theory. In lattice gauge theory, a manifest gauge invariant geometrical basis in the physical Hilbert space is given by the set of all possible Wilson loop states

$$|\Gamma\rangle = W(\Gamma)|0\rangle \tag{A.1}$$

$W(\Gamma)$ is the product of link operators along any oriented, closed-loop Γ and $|0\rangle$ is the strong coupling vacuum. The problem with the above loop basis is that it is overcomplete. The Mandelstam constraints [2, 41, 49, 50, 52–54, 57, 67, 71, 73–75, 89, 129–131, 136, 147] amongst the various loop states express this overcompleteness of the Wilson loop basis. The Mandelstam constraints are relations between Wilson loops which reflect the structure of the gauge group. Mandelstam constraints on a lattice For an SU(N) gauge theory on a lattice, there are $\mathrm{N}^2 - 1$ degrees of freedom on each link and $\mathrm{N}^2 - 1$ Gauss law constraints on each site. Therefore, the number of gauge invariant degrees of freedom N_{gidf} is given by the dimension of the quotient space $\otimes_{links} SU(\mathrm{N}) / \otimes_{sites} SU(\mathrm{N})$. On a d dimensional lattice with N_s sites and $N_s - 1$ links along any direction and open boundary condition, it is given by

$$N_{gidf} = (\mathrm{N}^2 - 1)(\mathcal{L} - \mathcal{N}) = (\mathrm{N}^2 - 1)\left[d(N_s - 1)(N_s)^{d-1} - (N_s)^d\right] \tag{A.2}$$

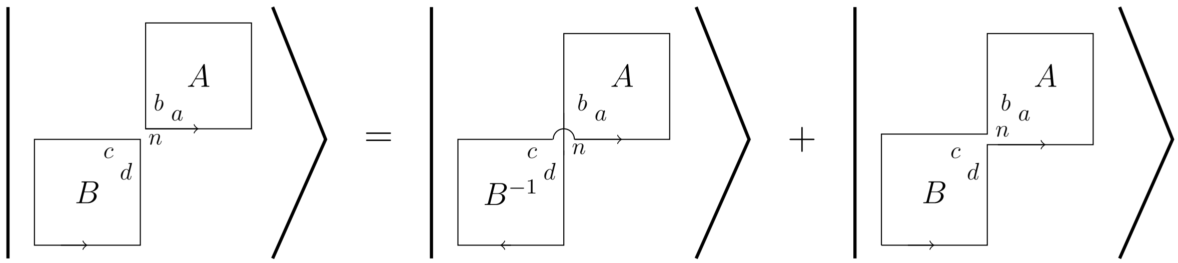


Figure A.1: Simplest example of Mandelstam constraints for SU(2) gauge theory on a two dimensional spatial lattice

in above, $\mathcal{L}$ and $\mathcal{N}$ represents the total no of links and sites on the lattice. But, there is an infinite number of Wilson loops even on a finite lattice. Therefore, the Wilson loop basis is clearly over-complete. A classical equivalence theorem [57] states that once the Mandelstam constraints are solved, the number of independent loop variables left would equal the number of physical degrees of freedom. Therefore, the extra loop degrees of freedom have to be removed by imposing the Mandelstam constraints. The origin of Mandelstam constraints can be traced back to the identities satisfied by the traces of SU (N) matrices. For concreteness, we consider SU(2) gauge theory on a 2-dimensional lattice for illustrating the Mandelstam constraints. In order to describe a simple example of Mandelstam constraints, let's consider 2 plaquettes A and B touching each other at a common lattice site n as shown in Figure A.1. The corresponding Wilson loop operators satisfy [129–131, 136]

$$\mathrm{Tr}(U_A)\mathrm{Tr}(U_B) = \mathrm{Tr}(U_A U_B) + \mathrm{Tr}(U_A U_B^{-1}) \tag{A.3}$$

The above relation is a trivial identity involving any two SU(2) matrices U_A and U_B. It can be easily checked by writing the SU(2) matrices in the following representation: $U_X = x_0 I + i\sigma^{\mathrm{a}} X_{\mathrm{a}}$, where σ_{a} a are the pauli matrices, (X_0, X_{a}) are real and satisfy $X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1$. There fore $U_A = A_0 + i\sigma^{\mathrm{a}} A_{\mathrm{a}}$ and $U_B = B_0 + i\sigma^{\mathrm{a}} B_{\mathrm{a}}$ and $\mathrm{Tr}(U_A U_B) = 2A_0 B_0 - 2A_{\mathrm{a}} B_{\mathrm{a}}$ and $\mathrm{Tr}(U_A U_B^{-1}) = 2A_0 B_0 + 2A_{\mathrm{a}} B_{\mathrm{a}}$ and $\mathrm{Tr}(U_A)\mathrm{Tr}(U_B) = 4A_0 B_0$. This leads to the above identity. The above operator identity implies the following relation between the corresponding loop states

$$|\gamma_1\rangle = \mathrm{Tr}(U_A)\mathrm{Tr}(U_B)|0\rangle, \quad |\gamma_2\rangle = \mathrm{Tr}(U_A U_B)|0\rangle, \quad |\gamma_3\rangle = \mathrm{Tr}(U_A U_B^{-1})|0\rangle \tag{A.4}$$

$$|\gamma_1\rangle = |\gamma_2\rangle + |\gamma_3\rangle \tag{A.5}$$

Thus, the loop states $|\gamma_1\rangle$, $|\gamma_2\rangle$, $|\gamma_3\rangle$ are linearly dependent. The Mandelstam constraints become more and more complicated when larger loops and loops of large fluxes are involved. To appreciate this better, let's consider most general loop states involving only these 2 plaquettes A and B [62, 137].

$$
\begin{aligned}
|N_A, N_B\rangle =&(\mathrm{Tr}(U_A))^{N_A}(\mathrm{Tr}(U_B))^{N_B}|0\rangle \\
=&(\mathrm{Tr}(U_A))^{N_A-1}(\mathrm{Tr}(U_B))^{N_B-1}(\mathrm{Tr}(U_AU_B)+\mathrm{Tr}(U_AU_B^{-1}))|0\rangle \\
=&(\mathrm{Tr}(U_A))^{N_A-2}(\mathrm{Tr}(U_B))^{N_B-2}(\mathrm{Tr}(U_AU_B)+\mathrm{Tr}(U_AU_B^{-1}))^2|0\rangle \\
=&\cdots \\
=&(\mathrm{Tr}(U_A))^{N_A-N_{min}}(\mathrm{Tr}(U_B))^{N_B-N_{min}}(\mathrm{Tr}(U_AU_B)+\mathrm{Tr}(U_AU_B^{-1}))^{N_{min}}|0\rangle
\end{aligned}
\tag{A.6}
$$

where N_A, N_B are two arbitrary integers giving the fluxes over the plaquettes A and B and N_{min} = Minimum(N_A, N_B). Thus, the above expression gives $2N_{min}+1$ different linearly dependant loop states. The number of loop states and the constraints between them increases with the SU(2) flux value N_{min}. Adding more plaquettes gives more complicated loop states as well as Mandelstam constraints. It is also clear that this problem becomes worse in higher dimensions and higher gauge groups. In the strong coupling $g^2 \to \infty$ expansion technique, in low orders of perturbation theory, one deals with only a finite number of small loops with small fluxes. Therefore, Mandelstam constraints can be easily solved using Gram-Schmidt orthogonalisation techniques. But, in going to the weak coupling con- continuum limit $g^2 \to \infty$, large loops with large fluxes become important and Mandelstam constraints become more and more involved. Therefore, one is confronted with the problem of finding a complete set of linearly independent loop states amongst loop states of all shapes, sizes and carrying arbitrary fluxes and touching/crossing each other at arbitrary lattice sites.

APPENDIX B

CANONICAL TRANSFORMATIONS ON 2×2 PLAQUETTE LATTICE

In this Appendix, we will explicitly work out SU(N) canonical transformations (5.6), (5.12a) and (5.12b) for the simple 2×2 plaquette lattice. Starting from the top-left plaquette, we make canonical transformations over four plaquettes in the following four steps I, II, III and IV to construct the plaquettes $\mathcal{W}(0,1)$, $\mathcal{W}(0,0)$, $\mathcal{W}(1,1)$ and $\mathcal{W}(1,0)$ respectively (see Figure B.5). Each of these 4 steps involves 3 gluings of Kogut-Susskind link holonomies through canonical transformations illustrated in Figure 5.2.

I. Construction of $\mathcal{W}(0,1)$:

In the first step we glue four links of the top-left plaquette in the clockwise direction and convert them into the plaquette $\mathcal{W}(0,1)$ and the three remaining holonomies: $\mathcal{U}(0,1;\hat{2})$, $\tilde{\mathcal{U}}(1,1;\hat{2})$, $\tilde{\mathcal{U}}(0,1;\hat{1})$ as shown in Figure B.5-I. The three canonical transformations involved in this first step are:

1. The first canonical transformation, shown in Figure B.1, is:

$$\begin{bmatrix} (E_+(0,1;\hat{2}),\, U(0,1;\hat{2})) \\ \\ (E_+(0,2;\hat{1}),\, U(0,2;\hat{1})) \end{bmatrix} \rightarrow \begin{bmatrix} (\mathcal{E}_+(0,1;\hat{2}),\, \mathcal{U}(0,1;\hat{2})) \\ \\ (\tilde{\mathcal{E}}_+(0,1),\, \tilde{\mathcal{U}}(0,1)) \end{bmatrix}$$

The two new holonomies are defined as

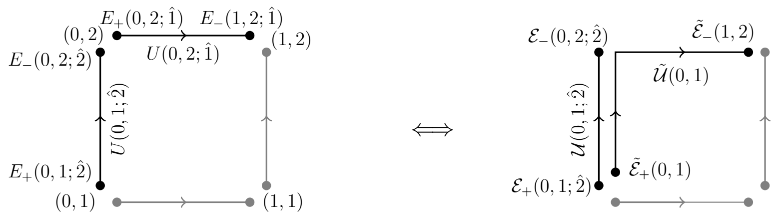


Figure B.1: First canonical transformation (see equations, (B.1), (B.2), (B.3) and (B.4)) over the top leftmost plaquette for 2 × 2 lattice. The canonical transformation provides two new electric fields $\mathcal{E}_-(0,2;\hat{2})$ and $\tilde{\mathcal{E}}_-(1,2)$. Other two electric fields $\mathcal{E}_+(0,2;\hat{2})$ and $\tilde{\mathcal{E}}_+(0,1)$ are obtained using parallel transports in equations B.5 and (B.6) respectively. In this canonical transformation, we have obtained a dual string $\mathcal{U}(0,1;\hat{2})$ and an intermediate holonomy $\tilde{\mathcal{U}}(0,1)$ which is used in second canonical transformation (see Figure B.2).

$$\mathcal{U}(0,1;\hat{2}) = U(0,1;\hat{2}) \tag{B.1}$$
$$\tilde{\mathcal{U}}(0,1) = U(0,1;\hat{2})U(0,2;\hat{1}). \tag{B.2}$$

The basic canonical transformations (5.1) determine their right electric fields

$$\mathcal{E}_-(0,2;\hat{2}) = E_-(0,2;\hat{2}) + E_+(0,2;\hat{1}) \tag{B.3}$$
$$\tilde{\mathcal{E}}_-(1,2) = E_-(1,2;\hat{1}). \tag{B.4}$$

In the above equations, $\mathcal{E}_-(0,2;\hat{2})$ and $\tilde{\mathcal{E}}_-(1,2)$ are the right electric fields of $\mathcal{U}(0,1;\hat{2})$ and $\tilde{\mathcal{U}}(0,1)$ respectively and reside at sites $(0,2)$ and $(1,2)$ respectively. The canonical transformations (B.1), (B.2), (B.3), (B.4) and the electric field locations are shown in Figure B.1. Their corresponding left electric fields can be obtained by parallel transports as in (3.14):

$$\begin{aligned}\mathcal{E}_+(0,1;\hat{2}) &= -\,\mathcal{U}(0,1;\hat{2})\mathcal{E}_-(0,2;\hat{2})\mathcal{U}^\dagger(0,1;\hat{2}) \\ &= E_+(0,1;\hat{2}) + U(0,1;\hat{2})U(0,2;\hat{1})\ E_-(1,2;\hat{1})\ U^\dagger(0,2;\hat{1})U^\dagger(0,1;\hat{2}) \\ &= E_+(0,1;\hat{2}) + \mathcal{S}_2(0,1)\ E_-(1,2;\hat{1})\ \mathcal{S}_2^{-1}(0,1)\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}\tilde{\mathcal{E}}_+(0,1) &= -\,\tilde{\mathcal{U}}(0,1)\ E_-(1,2;\hat{1})\ \tilde{\mathcal{U}}^\dagger(0,1) \\ &= -\,\mathcal{S}_2(0,1)\ E_-(1,2;\hat{1})\ \mathcal{S}_2^{-1}(0,1)\end{aligned} \tag{B.6}$$

In (B.5) and (B.6) we have identified the parallel transport:

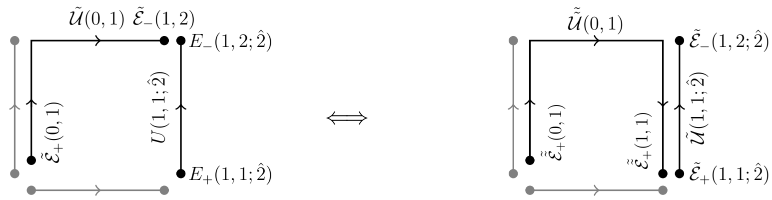


Figure B.2: Second canonical transformation (representing equations, (B.8), (B.9), (B.10)and (B.11)) over the top leftmost plaquette for 2 × 2 lattice.

$$\mathcal{S}_2(0,1) \equiv \tilde{\mathcal{U}}(0,1) = U(0,1;\hat{2})U(0,2;\hat{1}) \tag{B.7}$$

as defined in equation (5.7c) and in Figure 5.4-c. We thus obtain (5.12b) at $m = 0, n = 1$.

2. The second canonical transformation, shown in Figure B.2 is:

$$\begin{bmatrix} (\tilde{\mathcal{E}}_+(0,1),\ \tilde{\mathcal{U}}(0,1)) \\ \\ (E_+(1,1;\hat{2}),\ U(1,1;\hat{2})) \end{bmatrix} \rightarrow \begin{bmatrix} (\tilde{\tilde{\mathcal{E}}}_+(0,1),\ \tilde{\tilde{\mathcal{U}}}(0,1)) \\ \\ (\tilde{\mathcal{E}}_+(1,1;\hat{2}),\ \tilde{\mathcal{U}}(1,1;\hat{2})) \end{bmatrix}$$

The two new holonomies are defined as

$$\tilde{\mathcal{U}}(1,1;\hat{2}) = U(1,1;\hat{2}) \tag{B.8}$$

$$\tilde{\tilde{\mathcal{U}}}(0,1) = \tilde{\mathcal{U}}(0,1)\ U^\dagger(1,1;\hat{2}) = \ U(0,1;\hat{2})U(0,2;\hat{1})U^\dagger(1,1;\hat{2}) \tag{B.9}$$

The canonical transformations (5.1) lead to the following electric fields

$$\tilde{\mathcal{E}}_-(1,2;\hat{2}) = E_-(1,2;\hat{2}) + \tilde{\mathcal{E}}_-(0,1) = \ E_-(1,2;\hat{2}) + E_-(1,2;\hat{1}) \tag{B.10}$$

$$\tilde{\tilde{\mathcal{E}}}_+(0,1) = \tilde{\mathcal{E}}_+(0,1) = \ -\mathcal{S}_2(0,1)E_-(1,2;\hat{1})\mathcal{S}_2^{-1}(0,1) \tag{B.11}$$

In equation (B.10), $\tilde{\mathcal{E}}_-(1,2;\hat{2})$ is the right electric field of intermediate holonomy $\tilde{\mathcal{U}}(1,1;\hat{2})$. The canonical transformations (B.8), (B.9), (B.10), (B.11) and the electric field locations are shown in Figure B.2. We now use (3.14) and obtain the left and right electric fields of $\tilde{\mathcal{U}}(1,1;\hat{2})$ and $\tilde{\tilde{\mathcal{U}}}(0,1)$ respectively for later use (in equations (B.17) and (B.34) respectively),

$$\begin{aligned} \tilde{\mathcal{E}}_+(1,1;\hat{2}) &= -\mathcal{U}(1,1;\hat{2})\tilde{\mathcal{E}}_-(1,1;\hat{2})\mathcal{U}^\dagger(1,1;\hat{2}) \\ &= E_+(1,1;\hat{2}) - U(1,1;\hat{2})E_-(1,2;\hat{1})U^\dagger(1,1;\hat{2}) \\ &= E_+(1,1;\hat{2}) - \mathcal{S'}_2(1,1)E_-(1,2;\hat{1})\mathcal{S'}_2^{-1}(1,1) \end{aligned} \tag{B.12}$$

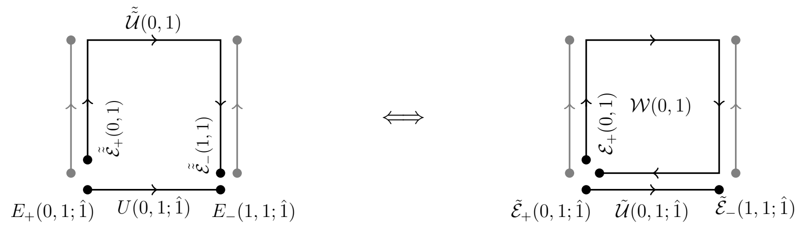


Figure B.3: Third canonical transformation (representing equations, (B.15), (B.16), (B.17), (B.18)) over the top leftmost plaquette for 2×2 lattice.

and

$$\begin{aligned}\tilde{\tilde{\mathcal{E}}}_-(1,1) &= -\tilde{\tilde{\mathcal{U}}}^\dagger(0,1)\tilde{\tilde{\mathcal{E}}}_+(0,1)\tilde{\tilde{\mathcal{U}}}(0,1) \\ &= {\mathcal{S}'}_2(1,1)E_-(1,2;\hat{1}){\mathcal{S}'}_2^{-1}(1,1)\end{aligned} \tag{B.13}$$

In (B.13) we have identified

$$ {\mathcal{S}'}_2(1,1) \equiv U(1,1;\hat{2}) \tag{B.14}$$

as defined in the (5.13c) and in Figure 5.4-f for $m = 1$.

3. The third canonical transformation, shown in Figure B.3 is

$$\begin{bmatrix} (\tilde{\tilde{\mathcal{E}}}_+(0,1),\ \tilde{\tilde{\mathcal{U}}}(0,1)) \\ \\ (E_+(0,1;\hat{1}),\ U(0,1;\hat{1})) \end{bmatrix} \rightarrow \begin{bmatrix} (\mathcal{E}_+(0,1),\mathcal{U}(0,1)) \\ \\ (\tilde{\mathcal{E}}_+(0,1;\hat{1}),\ \tilde{\mathcal{U}}(0,1;\hat{1})) \end{bmatrix}$$

We now glue the conjugate pair $(\tilde{\tilde{\mathcal{E}}}_+(0,1),\ \tilde{\tilde{\mathcal{U}}}(0,1))$ obtained in the previous step with link conjugate pair $(E_+(0,1;\hat{1}),\ U(0,1;\hat{1}))$ to get the first plaquette conjugate pair $(\mathcal{E}_+(0,1),\mathcal{W}(0,1))$ along with the intermediate link conjugate pair $(\tilde{\mathcal{E}}_+(0,1;\hat{1}),\tilde{\mathcal{U}}(0,1;\hat{1}))$

$$\tilde{\mathcal{U}}(0,1;\hat{1}) = U(0,1;\hat{1}), \tag{B.15}$$

$$\begin{aligned}\mathcal{W}(0,1) &= \tilde{\tilde{\mathcal{U}}}(0,1)\ U^\dagger(0,1;\hat{1}) \\ &= U(0,1;\hat{2})U(0,2;\hat{1})U^\dagger(1,1;\hat{2})U^\dagger(0,1;\hat{1})\end{aligned} \tag{B.16}$$

Their conjugate electric fields are

$$\tilde{\mathcal{E}}_-(1,1;\hat{1}) = E_-(1,1;\hat{1}) + \tilde{\tilde{\mathcal{E}}}_-(1,1)$$

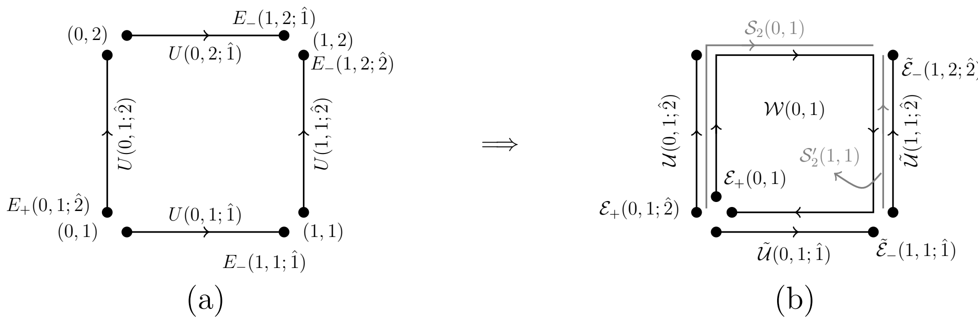


Figure B.4: The resulting degrees of freedom in I$^{\text{st}}$ step of canonical transformation. (a) Shows the initial degrees for freedom and their electric field used in the right-hand side of equations (B.19) and (B.20). (b) Show new degrees of freedom obtained in this step that appear on the left-hand side of equations (B.19) and (B.20). The strings $\mathcal{S}_2(0,1)$ and $\mathcal{S}'_2(1,1)$ are shown in gray.

$$= E_-(1,1;\hat{1}) + \mathcal{S}'_2(1,1)E_-(1,2;\hat{1})\mathcal{S}'^{-1}_2(1,1) \tag{B.17}$$

$$\mathcal{E}_+(0,1) = \tilde{\tilde{\mathcal{E}}}_+(0,1) = -\mathcal{S}_2(0,1)E_-(1,2;\hat{1})\mathcal{S}_2^{-1}(0,1) \tag{B.18}$$

In equation (B.17), $\tilde{\mathcal{E}}_-(1,1;\hat{1})$ is the right electric field corresponding to $\tilde{\mathcal{U}}(0,1;\hat{1})$. The above canonical transformations are shown in figure B.3. We thus obtain (5.6) for $m = 0, n = 1$.

The above three canonical transformations complete step I. In summary, starting from the 4 link holonomies

$$\big(U(0,1;\hat{2}),\, U(0,2;\hat{1}),\, U(1,1;\hat{2}),\, U(0,1;\hat{1})\big)$$

we have obtained the following 4 equivalent holonomies

$$\begin{aligned}
\mathcal{U}(0,1;\hat{2}) &= U(0,1;\hat{2}),\\
\mathcal{W}(0,1) &= U(0,1;\hat{2})U(0,2;\hat{1})U^\dagger(1,1;\hat{2})U^\dagger(0,1;\hat{1}),\\
\tilde{\mathcal{U}}(1,1;\hat{2}) &= U(1,1;\hat{2}),\\
\tilde{\mathcal{U}}(0,1;\hat{1}) &= U(0,1;\hat{1})
\end{aligned} \tag{B.19}$$

The corresponding electric fields are

$$\begin{aligned}
\mathcal{E}_+(0,1;\hat{2}) &= E_+(0,1;\hat{2}) + \mathcal{S}_2(0,1)E_-(1,2;\hat{1})\mathcal{S}_2^{-1}(0,1)\\
\mathcal{E}_+(0,1) &= -\mathcal{S}_2(0,1)E_-(1,2;\hat{1})\mathcal{S}_2^{-1}(0,1)\\
\tilde{\mathcal{E}}_-(1,2;\hat{2}) &= E_-(1,2;\hat{2}) + E_-(1,2;\hat{1})\\
\tilde{\mathcal{E}}_-(1,1;\hat{1}) &= E_-(1,1;\hat{1}) + \mathcal{S}'_2(1,1)E_-(1,2;\hat{1})\mathcal{S}'^{-1}_2(1,1)
\end{aligned} \tag{B.20}$$

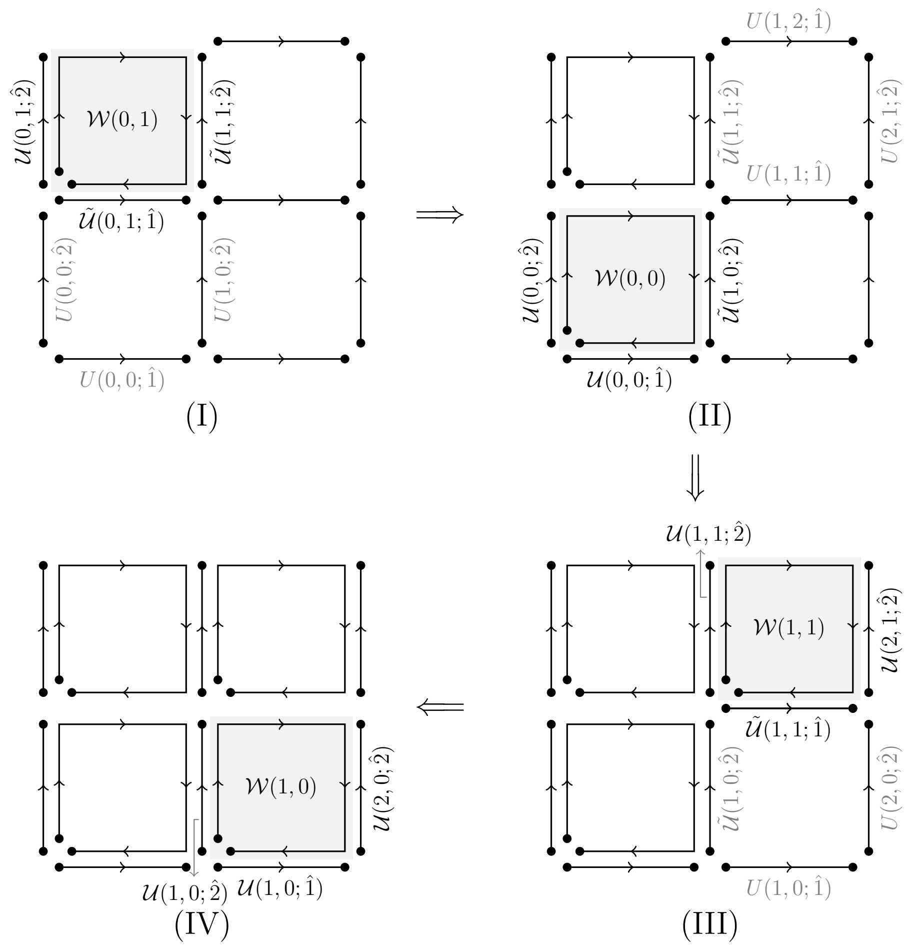


Figure B.5: The construction of $\mathcal{W}(0,1), \mathcal{W}(0,0), \mathcal{W}(1,1)$ and $\mathcal{W}(1,0)$ through canonical transformations in stages I, II, III and IV respectively. Each of these 4 stages involves 3 canonical transformations discussed in the text.

Now we notice that in step I, we have traded off $U(0,2;\hat{1})$ into the plaquette $\mathcal{W}(0,1)$ so its electric field $E_{-}(1,2;\hat{1})$ appears in all the four new degrees of freedom, plaquette & string electric fields with appropriate parallel transports (B.7) and (B.14). This step I is illustrated in Figure B.4. Now we will perform steps II, III and IV using equations (B.19) and (B.20). So we have two dual holonomies as required and two intermediate holonomies which will be used for canonical transformation in steps II and III. Electric fields for these holonomies are, see Figure B.4.

II. Construction of $\mathcal{W}(0,0)$:

In the second step, we consider four holonomies $U(0,0;\hat{2}), \tilde{\mathcal{U}}(0,1;\hat{1})$, $U(1,0;\hat{2})$ and $U(0,0;\hat{1})$,

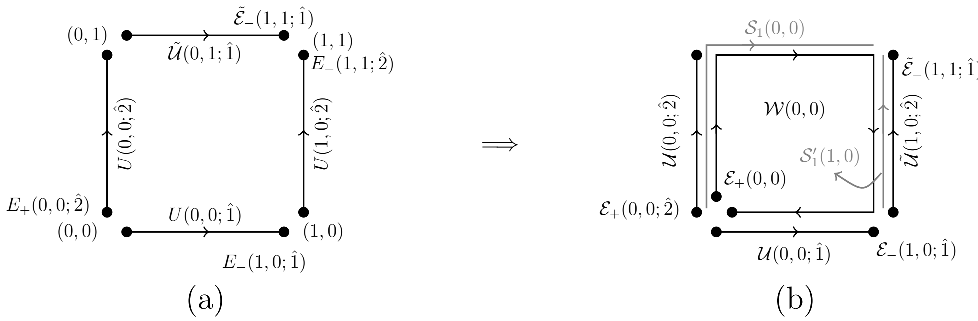


Figure B.6: The resulting degrees of freedom in II$^{\text{nd}}$ step of canonical transformation. (a) Shows the initial degrees for freedom and their electric field used in the right-hand side of equations (B.21) and (B.22). (b) Show new degrees of freedom obtained in this step that appear on the left-hand side of equations (B.21) and (B.22). The strings $\mathcal{S}_1(0,0)$ and $\mathcal{S}'_1(1,0)$ are shown in gray.

see Figure B.5-I and canonically convert them into following four holonomies see Figure B.5-II;

$$\begin{aligned}
\mathcal{U}(0,0;\hat{2}) =& U(0,0;\hat{2}),\\
\mathcal{W}(0,0) =& U(0,0;\hat{2})\tilde{\mathcal{U}}(0,1;\hat{1})U^\dagger(1,0;\hat{2})U^\dagger(0,0;\hat{1}),\\
=& U(0,0;\hat{2})U(0,1;\hat{1})U^\dagger(1,0;\hat{2})U^\dagger(0,0;\hat{1}),\\
\tilde{\mathcal{U}}(1,0;\hat{2}) =& U(1,0;\hat{2}),\\
\mathcal{U}(0,0;\hat{1}) =& U(0,0;\hat{1}),
\end{aligned} \tag{B.21}$$

with their electric fields that are given by

$$\begin{aligned}
\mathcal{E}_+(0,0;\hat{2}) =& E_+(0,0;\hat{2}) + \mathcal{S}_1(0,0)\tilde{\mathcal{E}}_-(1,1;\hat{1})\mathcal{S}_1^{-1}(0,0)\\
\mathcal{E}_+(0,0) =& -\mathcal{S}_1(0,0)\tilde{\mathcal{E}}_-(1,1;\hat{1})\mathcal{S}_1^{-1}(0,0)\\
\tilde{\mathcal{E}}_-(1,1;\hat{2}) =& E_-(1,1;\hat{2}) + \tilde{\mathcal{E}}_-(1,1;\hat{1})\\
\mathcal{E}_-(1,0;\hat{1}) =& E_-(1,0;\hat{1}) + \mathcal{S'}_1(1,0)\tilde{\mathcal{E}}_-(1,1;\hat{1})\mathcal{S'}_1^{-1}(1,0)
\end{aligned} \tag{B.22}$$

The above canonical transformation is shown in Figure B.6 and in (B.22) we have identified strings

$$\mathcal{S}_1(0,0) \equiv U(0,0;\hat{2})U(0,1;\hat{1}), \tag{B.23}$$

$$\mathcal{S'}_1(1,0) \equiv U(1,0;\hat{2}) \tag{B.24}$$

as defined in (5.7a) for $m = 0$ and (5.13a) for $m = 0$ respectively. Now we can use the expression of $\tilde{\mathcal{E}}_{-}(1,1;\hat{1})$ given in step I (equation (B.20)) to get following;

$$\mathcal{E}_{+}(0,0;\hat{2}) = E_{+}(0,0;\hat{2}) + \sum_{j=1}^{2} \mathcal{S}_j(0,0) E_{-}(1,j;\hat{1}) \mathcal{S}_j^{-1}(0,0) \tag{B.25}$$

$$\mathcal{E}_{+}(0,0) = -\sum_{j=1}^{2} \mathcal{S}_j(0,0) E_{-}(1,j;\hat{1}) \mathcal{S}_j^{-1}(0,0) \tag{B.26}$$

$$\tilde{\mathcal{E}}_{-}(1,1;\hat{2}) = E_{-}(1,1;\hat{2}) + E_{-}(1,1;\hat{1}) + \mathcal{S}'_2(1,0) E_{-}(1,2;\hat{1}) \mathcal{S}'^{-1}_2(1,0) \tag{B.27}$$

$$\mathcal{E}_{-}(1,0;\hat{1}) = E_{-}(1,0;\hat{1}) + \sum_{j=1}^{2} \mathcal{S}'_j(1,0) E_{-}(1,j;\hat{1}) \mathcal{S}'^{-1}_j(1,0) \tag{B.28}$$

In the above equations, we have identified strings

$$\mathcal{S}_2(0,0) \equiv U(0,0;\hat{2}) U(0,1;\hat{1}) U(1,1;\hat{2}), \tag{B.29}$$

$$\mathcal{S}'_2(1,0) \equiv U(1,0;\hat{2}) U(1,1;\hat{2}) \tag{B.30}$$

as defined in (5.7b) for $m = 0$ and (5.13b) for $m = 1$ respectively. Thus we have obtain (5.12b) and (5.6) for $m, n = 0$. We use (B.27) and (B.28) to obtain left electric fields for holonomies $\tilde{\mathcal{U}}(1,0;\hat{2})$ and $\mathcal{U}(0,0;\hat{1})$ respectively;

$$\tilde{\mathcal{E}}_{+}(1,0;\hat{2}) = E_{+}(1,0;\hat{2}) - \sum_{j=1}^{2} \mathcal{S}'_j(1,0) E_{-}(1,j;\hat{1}) \mathcal{S}'^{-1}_j(1,0) \tag{B.31}$$

$$\mathcal{E}_{+}(0,0;\hat{1}) = E_{+}(0,0;\hat{1}) - \sum_{j=1}^{2} \mathcal{S}_j(0,0) E_{-}(1,j;\hat{1}) \mathcal{S}_j^{-1}(0,0) \tag{B.32}$$

(B.31) will be used in step IV and (B.32) is (5.12a) for $m = 0$.

III. Construction of $\mathcal{W}(1,1)$:

In the third step, we start with four holonomies $\tilde{\mathcal{U}}(1,1;\hat{2})$, $U(1,2;\hat{1})$, $U(2,1;\hat{2})$ and $U(1,1;\hat{1})$ of top right plaquette see Figure B.5-II, and performing canonical transformations similar to previous step II we convert them into one plaquette, two strings and one intermediary holonomy;

$$\begin{aligned}
\mathcal{U}(1,1;\hat{2}) &= \tilde{\mathcal{U}}(1,1;\hat{2}) = U(1,1;\hat{2}), \\
\mathcal{W}(1,1) &= \tilde{\mathcal{U}}(1,1;\hat{2}) U(1,2;\hat{1}) U^{\dagger}(2,1;\hat{2}) U^{\dagger}(1,1;\hat{1}), \\
&= U(1,1;\hat{2}) U(1,2;\hat{1}) U^{\dagger}(2,1;\hat{2}) U^{\dagger}(1,1;\hat{1}), \\
\mathcal{U}(2,1;\hat{2}) &= U(2,1;\hat{2}), \\
\tilde{\mathcal{U}}(1,1;\hat{1}) &= U(1,1;\hat{1}),
\end{aligned} \tag{B.33}$$

above holonomies are shown in Figure B.5-III and their electric fields are given by

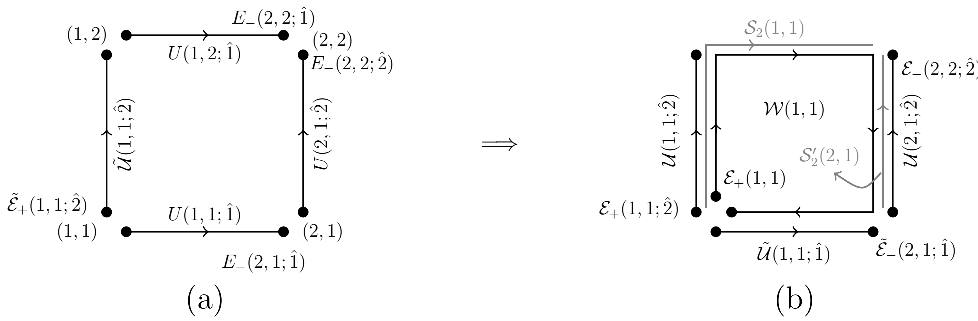


Figure B.7: The resulting degrees of freedom in III[st] step of canonical transformation. (a) Shows the initial degrees for freedom and their electric field used in the right-hand side of equations (B.33) and (B.34),(B.35),(B.36),(B.37). (b) Show new degrees of freedom obtained in this step that appear on the left-hand side of equations (B.33) and (B.34),(B.35),(B.36),(B.37). The strings $\mathcal{S}_2(1,1)$ and $\mathcal{S}_2'(2,1)$ are shown in gray.

$$\mathcal{E}_+(1,1;\hat{2}) = \tilde{\mathcal{E}}_+(1,1;\hat{2}) + \mathcal{S}_2(1,1)E_-(2,2;\hat{1})\mathcal{S}_2^{-1}(1,1) \tag{B.34}$$

$$\mathcal{E}_+(1,1) = -\mathcal{S}_2(1,1)E_-(2,2;\hat{1})\mathcal{S}_2^{-1}(1,1) \tag{B.35}$$

$$\mathcal{E}_-(2,2;\hat{2}) = E_-(2,2;\hat{2}) + E_-(2,2;\hat{1}) \tag{B.36}$$

$$\tilde{\mathcal{E}}_-(2,1;\hat{1}) = E_-(2,1;\hat{1}) + \mathcal{S}'_2(2,1)E_-(2,2;\hat{1})\mathcal{S}'^{-1}_2(2,1) \tag{B.37}$$

Canonical transformation in above equations, (B.34),(B.35),(B.36),(B.37) are showing in Figure B.7 and we have identified strings

$$\mathcal{S}_2(1,1) \equiv U(1,1;\hat{2})U(1,2;\hat{1}), \tag{B.38}$$

$$\mathcal{S}'_2(2,1) \equiv U(2,1;\hat{2}) \tag{B.39}$$

as defined in (5.7c) for $m=1$ and (5.13c) for $m=2$ respectively. Using (B.12) into (B.34) we obtain (5.12b) for $m=n=1$ and (B.35) is nothing but (5.6) for $m=n=1$. (B.36) is used to write left electric field of string $\mathcal{U}(2,1;\hat{2})$;

$$\mathcal{E}_+(2,1;\hat{2}) = E_+(2,1;\hat{2}) - \mathcal{S}'_2(2,1)E_-(2,2;\hat{1})\mathcal{S}'^{-1}_2(2,1) \tag{B.40}$$

which is (5.12b) for $m=2, n=1$ and (B.37) is used for canonical transformations in step IV.

IV. Construction of $\mathcal{W}(1,0)$:

In the fourth step we take four holonomies $\tilde{\mathcal{U}}(1,0;\hat{2})$, $\tilde{\mathcal{U}}(1,1;\hat{1})$, $U(2,0;\hat{2})$ and $U(1,0;\hat{1})$, see

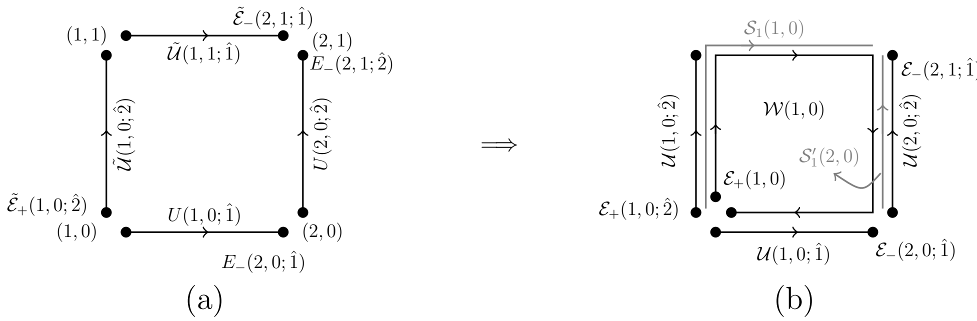


Figure B.8: The resulting degrees of freedom in IV$^{\text{nd}}$ step of canonical transformation. (a) Shows the initial degrees for freedom and their electric field used in the right-hand side of equations (B.41) and (B.42). (b) Show new degrees of freedom obtained in this step that appear on the left-hand side of equations (B.41) and (B.42). The strings $\mathcal{S}_1(1,0)$ and $\mathcal{S}_1'(2,0)$ are shown in gray.

Figure B.5-III and canonically convert them into following four holonomies, see Figure B.5-IV;

$$\begin{aligned} \mathcal{U}(1,0;\hat{2}) &= \tilde{\mathcal{U}}(1,0;\hat{2}) = U(1,0;\hat{2}), \\ \mathcal{W}(1,0) &= U(1,0;\hat{2})\tilde{\mathcal{U}}(1,1;\hat{1})U^\dagger(2,0;\hat{2})U^\dagger(1,0;\hat{1}), \\ &= U(1,0;\hat{2})U(1,1;\hat{1})U^\dagger(2,0;\hat{2})U^\dagger(1,0;\hat{1}), \\ \mathcal{U}(2,0;\hat{2}) &= U(2,0;\hat{2}), \\ \mathcal{U}(1,0;\hat{1}) &= U(1,0;\hat{1}), \end{aligned} \tag{B.41}$$

with their electric fields

$$\begin{aligned} \mathcal{E}_+(1,0;\hat{2}) =& \tilde{\mathcal{E}}_+(1,0;\hat{2}) + \mathcal{S}_1(1,0)\tilde{\mathcal{E}}_-(2,1;\hat{1})\mathcal{S}_1^{-1}(1,0) \\ \mathcal{E}_+(1,0) =& -\mathcal{S}_1(1,0)\tilde{\mathcal{E}}_-(2,1;\hat{1})\mathcal{S}_1^{-1}(1,0) \\ \mathcal{E}_-(2,1;\hat{2}) =& E_-(2,1;\hat{2}) + \tilde{\mathcal{E}}_-(2,1;\hat{1}) \\ \mathcal{E}_-(2,0;\hat{1}) =& E_-(2,0;\hat{1}) + \mathcal{S'}_1(2,0)\tilde{\mathcal{E}}_-(2,1;\hat{1})\mathcal{S'}_1^{-1}(2,0) \end{aligned} \tag{B.42}$$

Canonical transformations in the above equations (B.41) and (B.42) are shown in Figure B.8 and we have identified strings

$$\mathcal{S}_1(1,0) \equiv U(1,0;\hat{2})U(1,1;\hat{1}), \tag{B.43}$$

$$\mathcal{S'}_1(2,0) \equiv U(2,0;\hat{2}) \tag{B.44}$$

as defined in (5.7a) for $m = 1$ and (5.13a) for $m = 2$ respectively. Now we can use the expression of $\tilde{\mathcal{E}}_+(1,0;\hat{2})$ and $\tilde{\mathcal{E}}_-(2,1;\hat{1})$ obtained in the step II and III respectively;

$$\mathcal{E}_+(1,0;\hat{2}) = E_+(1,0;\hat{2}) - \sum_{j=1}^{2} \mathcal{S}'_j(1,0) E_-(1,j;\hat{1}) \mathcal{S}'^{-1}_j(1,0) + \sum_{j=1}^{2} \mathcal{S}_j(1,0) E_-(2,j;\hat{1}) \mathcal{S}^{-1}_j(1,0) \tag{B.45}$$

$$\mathcal{E}_+(1,0) = -\sum_{j=1}^{2} \mathcal{S}_j(1,0) E_-(2,j;\hat{1}) \mathcal{S}^{-1}_j(1,0) \tag{B.46}$$

$$\mathcal{E}_-(2,1;\hat{2}) = E_-(2,1;\hat{2}) + E_-(2,1;\hat{1}) + \mathcal{S}'_2(2,0) E_-(2,2;\hat{1}) \mathcal{S}'^{-1}_2(2,0) \tag{B.47}$$

$$\mathcal{E}_-(2,0;\hat{1}) = E_-(2,0;\hat{1}) + \sum_{j=1}^{2} \mathcal{S}'_j(2,0) E_-(2,j;\hat{1}) \mathcal{S}'^{-1}_j(2,0) \tag{B.48}$$

In the above equations, we have identified strings

$$\mathcal{S}_2(1,0) \equiv U(1,0;\hat{2}) U(1,1;\hat{1}) U(2,1;\hat{2}), \tag{B.49}$$

$$\mathcal{S}'_2(2,0) \equiv U(2,0;\hat{2}) U(2,1;\hat{2}) \tag{B.50}$$

as defined in (5.7b) for $m = 0$ and (5.13b) for $m = 1$ respectively. (B.45) and (B.46) are (5.12b) and (5.6) for $m = 1, n = 0$ respectively. We use (B.47) and (B.48) to write left electric fields for holonomies $U(2,0;\hat{2})$ and $U(1,0;\hat{1})$ respectively.

$$\mathcal{E}_+(2,0;\hat{2}) = E_+(2,0;\hat{2}) - \sum_{j=1}^{2} \mathcal{S}'_j(2,0) E_-(2,j;\hat{1}) \mathcal{S}'^{-1}_j(2,0) \tag{B.51}$$

$$\mathcal{E}_+(1,0;\hat{1}) = E_+(1,0;\hat{1}) - \sum_{j=1}^{2} \mathcal{S}_j(1,0) E_-(2,j;\hat{1}) \mathcal{S}^{-1}_j(1,0) \tag{B.52}$$

The above equations (B.51) and (B.52) are (5.12b) for $m = 2, n = 0$ and (5.12a) for $m = 1, n = 0$ respectively. Hence canonical transformations are completed on a 2×2 lattice. Following the above procedure systematically these canonical transformations can be generalized for any $N \times N$ lattice.

The plaquette constraints

We now show that the new plaquette constraints (5.33) weakly commute with the Hamiltonian (5.41) and therefore remain preserved under time evolution. We define the following operator

$$\mathcal{C}_{\alpha\beta}(\vec{n}) \equiv \big(\mathcal{U}_{\rm p}(\vec{n}) - \mathcal{W}(\vec{n})\big)_{\alpha\beta} \tag{B.53}$$

where

$$\mathcal{U}_{\rm p}(\vec{n}) \equiv \big(\mathcal{U}(\vec{n};\hat{2})\mathcal{U}(\vec{n}+\hat{2};\hat{1})\mathcal{U}^{\dagger}(\vec{n}+\hat{1};\hat{2})\mathcal{U}^{\dagger}(\vec{n};\hat{1})\big)\,.$$

Using (5.36) it is easy to prove that the Kogut-Susskind electric fields rotate $\mathcal{C}_{\alpha\beta}$ from left and right as follows

$$[E^{\rm a}_{+}(\vec{n}';\hat{1}),\ \mathcal{C}_{\alpha\beta}(\vec{n})] \;=\; \delta_{\vec{n}',\,\vec{n}}\,(\mathcal{C}(\vec{n})\;T^{\rm a})_{\alpha\beta} + \delta_{\vec{n}',\,\vec{n}+\hat{2}}\;R^{\rm ab}(\mathcal{U}^{\dagger}(\vec{n};\hat{2}))\;\big(T^{\rm b}\;\mathcal{C}(\vec{n})\big)_{\alpha\beta} \;\approx\; 0, \quad \text{(B.54a)}$$

$$[E^{\rm a}_{+}(\vec{n}';\hat{2}),\ C_{\alpha\beta}(\vec{n})] \;=\; \delta_{\vec{n}',\,\vec{n}}\,(T^{\rm a}\;\mathcal{C}(\vec{n}))_{\alpha\beta} + \delta_{\vec{n}',\,\vec{n}+\hat{2}}\;R^{\rm ab}(\mathcal{U}^{\dagger}(\vec{n};\hat{1}))\;\big(\mathcal{C}(\vec{n})\;T^{\rm b}\big)_{\alpha\beta} \;\approx\; 0. \quad \text{(B.54b)}$$

In (B.54a) and (B.54b) we have used the plaquette, string sectors canonical commutation relations (5.8) and (5.14). They show that on the constrained surface the dual Hamiltonian commutes with the plaquette constraints:

$$[H, C_{\alpha\beta}(\vec{n})] \;\approx\; 0. \tag{B.55}$$

We also check the commutation relations of the constraints $C_{\alpha\beta}(\vec{n}) = 0$ with the Gauss law constraints (5.40):

$$[\mathcal{G}^{\rm a}(\vec{n}'), C_{\alpha\beta}(\vec{n})] = -\delta_{\vec{n}',\,\vec{n}}\;\,[T^{\rm a},\mathcal{C}(\vec{n})]_{\alpha\beta} \;\approx\; 0. \tag{B.56}$$

Therefore, the plaquette constraints (5.33) together with the SU(N) Gauss law constraints (5.40) define the physical Hilbert space where the dual loop dynamics with inverted coupling is local.

APPENDIX C

ELECTRIC LOOP BASIS

In this Appendix, we define the SU(N) electric plaquette loop basis in $\mathcal{H}^{\mathrm{p}hys}$. These basis vectors are the eigenvectors of (N − 1) electric field operators. They span the entire Hilbert space $\mathcal{H}^{\mathrm{p}hys}$ and get translated under the actions of Wilson loops or SU(N) order operators. It is easy to construct the loop basis in terms of the dual electric scalar potentials on the plaquette loops (see Figure 6.2). In the prepotential representation

$$\mathcal{E}^{\mathrm{a}}_{-}(p) \equiv a^{\dagger}(p)\,\frac{\sigma^{\mathrm{a}}}{2}\,a(p), \qquad \mathcal{E}^{\mathrm{a}}_{+}(p) \equiv -b(p)\,\frac{\sigma^{\mathrm{a}}}{2}\,b^{\dagger}(p). \tag{C.1}$$

Using the facts that the left and the right electric fields are independent, $[\mathcal{E}^{\mathrm{a}}_{+}(p),\ \mathcal{E}^{\mathrm{b}}_{-}(p)] = 0$, and their magnitudes are equal, $\sum_{\mathrm{a}=1}^{3} \mathcal{E}^{\mathrm{a}}_{+}\mathcal{E}^{\mathrm{a}}_{+} = \sum_{\mathrm{a}=1}^{3} \mathcal{E}^{\mathrm{a}}_{-}\mathcal{E}^{\mathrm{a}}_{-} \equiv \mathcal{E}^{2}$, we define the first set of a complete set of commuting operators on every plaquette p as: $\left[\mathcal{E}^{2},\ \mathcal{E}^{\mathrm{a}=3}_{+},\ \mathcal{E}^{\mathrm{a}=3}_{-}\right]$. The SU(2) electric loop decoupled basis on every plaquette p is

$$|j\ m_{+}\ m_{-}\rangle \equiv |j\ m_{+}\rangle \otimes |j\ m_{-}\rangle\ = \phi^{j}_{m_{+}}(a^{\dagger}_{1}, a^{\dagger}_{2})|0\rangle_{a} \otimes \phi^{j}_{m_{-}}(b^{\dagger}_{1}, b^{\dagger}_{2})|0\rangle_{b}, \tag{C.2}$$

where we have defined

$$\phi^{j}_{m}(a^{\dagger}_{1}, a^{\dagger}_{2}) = \frac{(a^{\dagger}_{1})^{j+m}}{\sqrt{(j+m)!}}\frac{(a^{\dagger}_{2})^{j-m}}{\sqrt{(j-m)!}}. \tag{C.3}$$

Under SU(2) gauge transformations at the origin $\Lambda = \Lambda(0,0)$

$$\begin{aligned} \phi^j_{m_+}(a_1^\dagger, a_2^\dagger) &\rightarrow D^{\;j}_{m_+ m'_+}(\Lambda)\ \phi^j_{m'_+}(a_1^\dagger, a_2^\dagger) \\ \phi^j_{m_-}(b_1^\dagger, b_2^\dagger) &\rightarrow D^{\;j}_{m_- m'_-}(\Lambda^\dagger)\ \phi^j_{m'_-}(b_1^\dagger, b_2^\dagger) \end{aligned} \tag{C.4}$$

The electric flux states transform as

$$|j\ m_+\ m_-\rangle = \sum_{m'_+, m'_-} D^{\;j}_{m_+ m'_+}(\Lambda) D^{\;j}_{m_- m'_-}(\Lambda^\dagger)\ |j\ m'_+ m'_-\rangle. \tag{C.5}$$

At this stage, it is convenient to work with the coupled basis instead of the decoupled basis (C.5). We define the complete set of commuting operators (CSCO) on every plaquette as

$$\mathcal{E}^2 = \mathcal{E}_+^2 = \mathcal{E}_-^2,\ L^{\mathrm{a}} \equiv \mathcal{E}_+^{\mathrm{a}} + \mathcal{E}_-^{\mathrm{a}},\ L^{\mathrm{a}=3} \equiv \mathcal{E}_-^{\mathrm{a}=3} + \mathcal{E}_+^{\mathrm{a}=3}. \tag{C.6}$$

The loop coupled basis on every lattice plaquette $|n\ l\ m\rangle$ can be constructed as

$$|n\ l\ m\ \rangle = \ N_{nlm}\ (k_+)^{n-l-1}\ (L_-)^{l-m} (a_1^\dagger)^l (b_2^\dagger)^l\ |0,0\rangle \tag{C.7}$$

In (C.7) $k_+ \equiv a^\dagger \cdot b^\dagger \equiv \sum_{\alpha=1}^{2} a_\alpha^\dagger b_\alpha^\dagger$ and

$$N_{nlm} \equiv \sqrt{\frac{n(l+m)!}{(l-m)!\,(l!)^2 (n-l-1)!\,(m+l)!}}.$$

The corresponding eigenvalue equations are

$$\begin{aligned} \mathcal{E}^2\ |n\ l\ m\rangle_p &= \left(\frac{n^2-1}{4}\right)\ |n\, l\ m\rangle_p \\ \vec{L}^2\ |n\ l\ m\rangle_p &= l(l+1)\ |n\, l\ m\rangle_p \\ L^{\mathrm{a}=3}\ |n\ l\ m\rangle_p &= m\ |n\, l\ m\rangle_p \end{aligned} \tag{C.8}$$

In above $l = 0, 1, 2, \ldots, n-1$ and $m = -l, -l+1, \ldots, l$. Under gauge transformations at the origin $\Lambda = \Lambda(0,0)$, these states have much simpler transformation property

$$|n\ l\ m\ \rangle = \sum_{\bar{m}} D_{m\ \bar{m}}^{\ l}(\Lambda) |n\ l\ \bar{m}\rangle \tag{C.9}$$

In other words the principal (n) and the angular momentum (l) quantum numbers remain invariant.

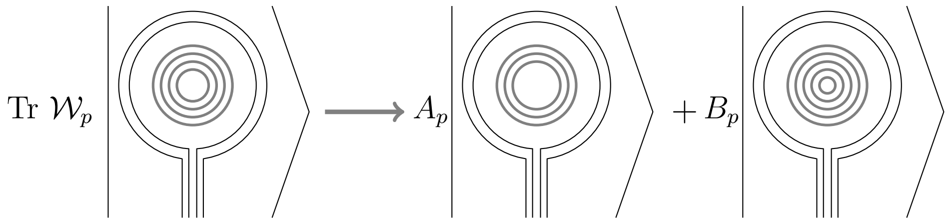


Figure C.1: The action of the Wilson loop on the loop state $|n = 4, l = 2, m\rangle$ described in the coupled basis. The circles in the three figures represent the SU(2) electric flux circulating in a loop within the plaquette and $2l$ is the number of open flux lines. The action of Tr $(\mathcal{W})$ simply translates n to $n \pm 1$ in (C.10).

The Wilson loops as translation operators

In the loop basis (C.8), the plaquette operators, $\mathcal{W}(p)$ which are unit size Wilson loop order operator acts as a translation operator for the electric flux n. Using (6.20) we get

$$\text{Tr}\ \mathcal{W}(p)\ |n\ l\ m\rangle = \ A_p\ |n-1\ l\ m\rangle + B_p\ |n+1\ l\ m\rangle. \tag{C.10}$$

Here we have ignored the plaquette index p on all three quantum numbers and

$$A_p = \frac{\sqrt{(n-l-1)(n+l)}}{(n-1)}, \qquad B_p = \frac{\sqrt{(n+l-1)(n-l)}}{(n+1)}.$$

The above translative action of the fundamental plaquette loop operator $\mathcal{W} \equiv \mathcal{W}(p)$ is valid on each plaquette p and we have suppresses the plaquette index p on both sides of (C.10). The action (C.10) is illustrated in Figure C.1. An arbitrary Wilson loop operator $\mathcal{W}(\mathcal{C})$ can be written in terms of the fundamental plaquette Wilson loop operators as

$$\mathcal{W}(\mathcal{C}) = \prod_{p\in\mathcal{C}} \mathcal{W}(p) = \mathcal{W}(p_1)\mathcal{W}(p_2)\cdots\mathcal{W}(p_{n_\mathcal{C}}). \tag{C.11}$$

The above product over plaquettes is taken from the bottom right corner as shown in Figure C.2. Therefore, the end effect of $\mathcal{W}(\mathcal{C})$ is to translate the electric fluxes of all plaquette loops inside the closed curve $\mathcal{C}$:

$$\mathcal{W}(\mathcal{C}) \prod_{p\in\mathcal{C}} |n\ l\ m\rangle_p = \prod_{p\in\mathcal{C}} \left(A_p\ |n-1\ l\ m\rangle_p + B_p\ |n+1\ l\ m\rangle_p\right). \tag{C.12}$$

Note that in the SU(N) case the Wilson loop operators will shift all the (N-1) Casimir eigen-

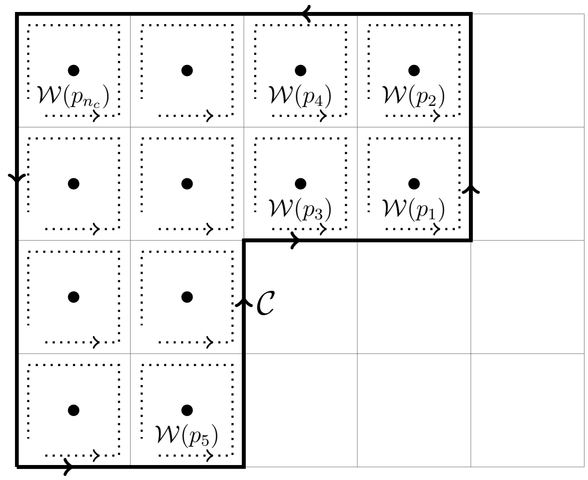


Figure C.2: The Wilson loop $\mathcal{W}(\mathcal{C})$ of any shape and size can be written as an ordered product of all plaquette operators $\mathcal{W}(p)$ inside $\mathcal{C}$ as in (C.11). These dotted plaquettes inside $\mathcal{C}$ are illustrated in Figure 6.2.

values by ± 1.

APPENDIX D

MAGNETIC LOOP BASIS

In this Appendix, we construct the magnetic basis for plaquette flux operators and show that the disorder operator has natural translative action on them. The group manifold for SU(2) group is S^3. We define it on every plaquette p through complex doublets $\vec{z}(p) \equiv (z_1(p), z_2(p))$ that satisfy the constraint $|z_1(p)|^2 + |z_2(p)|^2 = 1, \forall\, p$. A configuration on S^3 is

$$Z(p) = \begin{bmatrix} z_1(p) & z_2(p) \\ -z_2^*(p) & z_1^*(p) \end{bmatrix}. \tag{D.1}$$

We write eigenvalue equations for the magnetic flux operators as

$$\mathcal{W}_{\alpha\beta}(p)\ |Z(p)\rangle = Z_{\alpha\beta}(p)\ |Z(p)\rangle. \tag{D.2}$$

Here $Z_{\alpha\beta}(p)$ are the matrix elements of the matrix $Z(p)$ in (D.1). These states form a complete orthonormal basis on S^3

$$\int_{S^3} d\mu(\vec{z}\,)\ |Z(p)\rangle\langle Z(p)| = 1, \qquad \langle Z(p)|Z'(p)\rangle = \delta(Z(p) - Z'(p)). \tag{D.3}$$

The SU(2) group manifold integrations is defined as $\int_{SU(2)} d\mu(\vec{z}\,) \equiv \frac{1}{16\pi^2}\ \int d^2z_1 d^2z_2\, \delta(z_1^* z_1 + z_2^* z_2 - 1)$.

The magnetic eigenvectors $|Z(p)\rangle$ can be expanded in the complete orthonormal electric basis as

$$|Z(p)\rangle \equiv \ |z_1(p), z_2(p)\rangle = \sum_{j=0}^{\infty} \sqrt{(2j+1)} \sum_{m_+,m_-} D^{\ j}_{m_+\ m_-}(Z(p))\ |j\ m_+\ m_-\rangle$$

The construction of magnetic states can be easily checked by directly applying $\mathcal{W}$ on both sides above equations and realizing that $\mathcal{W}$ acts on the electric field basis as ladder and lowering operators and using the recurrence relations for the D-functions connecting $D^{\ j}_{m_+\ m_-}$ to $D^{\ j\pm\frac{1}{2}}_{m_+\pm\frac{1}{2}\ m_-\pm\frac{1}{2}}$. For SU(N), N $\geq$ 3, this approach gets extremely complicated as it requires the recurrence relations for the SU(N) Winger D-functions. We will first write down these states in terms of SU(2) prepotentials where they take a much simpler form and then verify the eigenvalues equations (D.2). Now use equation (C.2)

$$|Z(p)\rangle = \sum_{j=0}^{\infty} \sqrt{(2j+1)} \sum_{m_+,\,m_-} D^{\ j}_{m_+\ \ m_-}(Z(p))\ \phi^j_{m_+}(a^\dagger_1, a^\dagger_2)\ \phi^j_{m_-}(b^\dagger_1, b^\dagger_2)|0\rangle.$$

We call $\phi^j_m(x_1, x_2)$ the SU(2) structure functions. These SU(2) structure functions have the following orthonormal properties:

$$\begin{aligned} \int_{SU(2)} d\mu(\vec{z}\,)\ \phi^{j*}_m(z_1, z_2)\ \phi^j_{m'}(z_1, z_2) &= \frac{\delta_{m,m'}}{(2j+1)!}, \\ \sum_m \phi^{j*}_m(z_1, z_2)\phi^j_m(w_1, w_2) &= \frac{(z^*_1 w_1 + z^*_2 w_2)^{2j}}{(2j)!}. \end{aligned} \tag{D.4}$$

Further, we can also write SU(2) Wigner D-function in terms of these structure functions as follows:

$$D^j_{m,n}(z_1, z_2) = d_j \int_{SU(2)} d^2w_1 d^2w_2\ \phi^{j*}_m(w_1, w_2)\ \phi^j_n(z^w_1, z^w_2), \tag{D.5}$$

where

$$\begin{bmatrix} z^w_1 \\ z^w_2 \end{bmatrix} \equiv \begin{bmatrix} z_1 & z_2 \\ -z^*_2 & z^*_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad d_j \equiv (2j+1)$$

Using properties of structure functions and Wigner D functions,

$$\sum_{m_-=-j}^{j} D^j_{m_+\,m_-}(z_1, z_2)\ \phi^j_{m_-}(w_1, w_2\ ) = \ \phi^j_{m_+}(w^z_1, w^z_2)$$

in (D.4), we get

$$|Z(p)\rangle = \sum_{j=0}^{\infty} \sqrt{(2j+1)} \sum_{m_+} \phi^j_{m_+}(a_1^\dagger, a_2^\dagger)\ \phi^j_{m_+}(b_1^{z\dagger}, b_2^{z\dagger})|0\rangle,$$

where

$$\begin{bmatrix} b_1^{z\dagger} \\ b_2^{z\dagger} \end{bmatrix} \equiv \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix} \begin{bmatrix} b_1^\dagger \\ b_2^\dagger \end{bmatrix}.$$

Now we can sum the remaining magnetic index to get

$$|Z(p)\rangle = \sum_{j=0}^{\infty} \sqrt{d_j}\, \frac{\left(a^\dagger Z(p) b^\dagger\right)^{2j}}{(2j)!}\, |0,0\rangle, \tag{D.6}$$

where $d_j = (2j+1)$ is the dimension of $[j]$ representation. The eigenvalues equation (D.2) holds at each point of the group manifold. We first prove it for $Z = I$ where I is the identity element of SU(2) group. First, we prove that

$$\mathcal{W}^{j=1/2}_{\alpha\beta}\ |I\rangle = \delta_{\alpha\beta}\ |I\rangle. \tag{D.7}$$

where,

$$|I\rangle = \sum_{j=0}^{\infty} \frac{(2j+1)^{1/2}}{(2j)!} (a^\dagger \cdot b^\dagger)^{2j}\ |0,0\rangle \tag{D.8}$$

We have suppressed the plaquette index p. Using prepotential representation (6.20) for $\mathcal{W}_{\alpha\beta}$ we get

$$\mathcal{W}_{\alpha\beta}|I\rangle = \sum_{j=0}^{\infty} \frac{(2j+1)^{1/2}}{(2j)!} \left[\frac{a_\beta b_\alpha}{(2j)^{1/2}} - \frac{\tilde{a}^\dagger_\alpha \tilde{b}^\dagger_\beta}{(2j+2)^{1/2}}\right] \frac{1}{\sqrt{(2j+1)}} (a^\dagger \cdot b^\dagger)^{2j}\ |0\rangle$$

Now we replace $2j$ by $2j+1$ in the first term and $2j$ by $2j-1$ in the second term of above equation to get

$$\mathcal{W}_{\alpha\beta}|I\rangle = \sum_{j=0}^{\infty} \frac{(2j+1)^{1/2}}{(2j)!} \left[\frac{1}{(2j+1)^2} a_\beta b_\alpha\, (a^\dagger \cdot b^\dagger)^{2j+1} - \frac{(2j)}{(2j+1)} a^\dagger_\alpha b^\dagger_\beta (a^\dagger \cdot b^\dagger)^{2j-1}\right] |0\rangle$$

we evaluate first term using the prepotential commutation relations, $a_\alpha b_\beta (a^\dagger \cdot b^\dagger)^{2j+1}\,|0\rangle = a_\alpha b_\beta (\tilde{a}^\dagger \cdot \tilde{b}^\dagger)^{2j+1}|0\rangle = [(2j+1)^2 \delta_{\alpha\beta} (a^\dagger \cdot b^\dagger)^{2j} + (2j)(2j+1) \tilde{a}^\dagger_\alpha \tilde{b}^\dagger_\beta (a^\dagger \cdot b^\dagger)^{2j-1}\ ]|0\rangle$ and substitute in above equation to get:

$$\mathcal{W}_{\alpha\beta}|I\rangle = \sum_{j=0}^{\infty} \frac{(2j+1)^{1/2}}{(2j)!} \delta_{\alpha\beta} (a^\dagger \cdot b^\dagger)^{2j+1} |0\rangle = \delta_{\alpha\beta}|I\rangle$$

Now we can prove the eigenvalue equation (D.2), by considering a transformation of oscillators $b^\dagger_\alpha \rightarrow (Z^\dagger b^\dagger)_\alpha$. Under these transformations

$$\mathcal{W}_{\alpha\beta} \;\rightarrow\; Z^\dagger_{\alpha\gamma}\, \mathcal{W}_{\gamma\beta}, \qquad\qquad |I\rangle \rightarrow |Z\rangle$$

which yields

$$Z^\dagger_{\alpha\gamma} \mathcal{W}^{j=1/2}_{\gamma\beta}(p)\; |Z(p)\rangle = \; |Z(p)\rangle. \tag{D.9}$$

As $Z^\dagger Z = I$, we get the eigenvalue equation (D.2).

The conjugate electric fields act on this basis as differential operators on this plaquette holonomy basis.

$$\mathcal{E}^{\mathrm{a}}_{+}|Z\rangle = -\,\frac{\sigma^{\mathrm{a}}_{\alpha\beta}}{2}\, Z_{\gamma\beta}\, \frac{\partial}{\partial Z_{\alpha\gamma}}\, |Z\rangle \tag{D.10}$$

$$\mathcal{E}^{\mathrm{a}}_{-}\,|Z\rangle = \quad \frac{\sigma^{\mathrm{a}}_{\alpha\beta}}{2}\, Z_{\gamma\alpha}\, \frac{\partial}{\partial Z_{\gamma\beta}}\; |Z\rangle \tag{D.11}$$

For SU(3) these magnetic states are given by

$$|Z\rangle = \sum_{\mathrm{p,q}} \sqrt{d(\mathrm{p,q})}\; \frac{(a^\dagger[1] Z b^\dagger[1])^{\mathrm{p}}}{\mathrm{p}!}\; \frac{(a^\dagger[2] Z b^\dagger[2])^{\mathrm{q}}}{\mathrm{q}!} |0\rangle \tag{D.12}$$

$d(\mathrm{p,q}) = \frac{1}{2}(\mathrm{p}+1)(\mathrm{q}+1)(\mathrm{p}+\mathrm{q}+2)$ is dimension of [p, q] representation. For the general SU(N) case, these magnetic states are given as:

$$|Z\rangle \,=\, \sum_{[\vec{j}]=0}^{\infty} \sqrt{d(\vec{j})}\, \prod_{h=1}^{\mathrm{N}-1} \frac{1}{j_h!} \left(a^\dagger[h]\;\, Z\;\, b^\dagger[h]\right)^{2j_h}\; |0\rangle \tag{D.13}$$

Where $d(\vec{j})$ is the dimension of the $[\vec{j}]$ representation and Z represents (N $\times$ N) SU(N) matrix.

APPENDIX E

INVISIBILITY OF DIRAC STRING

In this Appendix, we show that only the endpoints of the Dirac strings are physical where they create the non-Abelian magnetic vortices [78]. We also show that in between the two endpoints, the Dirac strings can be moved around by the non-Abelian gauge transformations. These results also establish the algebra (7.46). We start with the disorder operator in Figure 7.5 with a straight horizontal Dirac string of length $\mathsf{L}+1$. The disorder operator is

$$\Sigma_{\vec{\omega}}(p_2, p_1) = \exp\left(i\,\omega \sum_{s=0}^{\mathsf{L}} \hat{\omega}_s^{\mathrm{a}} \cdot E_+^{\mathrm{a}}(s;\hat{2})\right), \tag{E.1}$$

where $\omega = \omega(x_0, y_0) = \omega_{s=0}$ and $E_+^{\mathrm{a}}(s;\hat{2}), s = 0, 1, \ldots, \mathsf{L}$ are the left or bottom electric fields on the vertical links $U(s\,;\hat{2}) \equiv U(x_0 + s, y_0;\, \hat{2})$ which generate SU(2) rotations. The axes of rotations $\hat{\omega}_s$ are related to $\hat{\omega}_{s=0} \equiv \omega(x_0, y_0) \equiv \hat{\omega}$. by parallel transports

$$\hat{\omega}_{h+1}^{\mathrm{a}} = R^{\mathrm{ab}}(\mathcal{W}^{\dagger}(h))U(h;\hat{1}))\ \hat{\omega}_h^{\mathrm{b}}, \quad h = 0, 1, \cdots, \mathsf{L}-1. \tag{E.2}$$

Here $\mathcal{W}(s) \equiv \mathcal{W}(x_0 + s, y_0)$ with the convention defined in (7.8) and $U(s,\, \hat{1}) \equiv U(x_0 + s, y_0;\, \hat{1})$. The parallel transports in the recurrence relations (E.2) are the underlying reason for the invisibility of the Dirac strings. This can be seen as follows. The $\mathsf{L}+1$ operators E^{a} in (E.1) rotate the corresponding $\mathsf{L}+1$ vertical links $U(s;\hat{2}), 0 \leq s \leq \mathsf{L}$ by Wigner D matrices as

$$\Sigma_{\vec{\omega}}(p_2, p_1)\; U_{\alpha\beta}(s\,;\,\hat{2})\, \Sigma_{\vec{\omega}}^{-1}(p_2, p_1) = \left(D^{j=\frac{1}{2}}(\hat{\omega}_s, \omega) U(s\,;\,\hat{2})\right)_{\alpha\beta} \tag{E.3}$$

In the above calculation, we have used the canonical commutation relations (3.12b). The set of rotated links perpendicular to the Dirac string $\mathcal{S}$ in Figure 7.5 do not affect the magnetic fields of any plaquette in the middle of the string but create magnetic vortices at the two end plaquettes p_1 and p_2. Note that any plaquette in the middle gets its two vertical links rotated around two different axes of rotations. These two axes are related by parallel transports in (E.2) so that the net effect or rotation is zero and therefore there are no vortex excitations. On the other hand, the two end plaquettes p_1 and p_2 of the Dirac string $\mathcal{S}$ have only one rotated link each. This results in the creation of two magnetic vortices at p_1, p_2. We now consider each of the above three cases separately.

case-I

We first consider the plaquette $\mathcal{W}(p_1)$. For the sake of calculations, we write $\mathcal{W} = U_B U_R U_T^\dagger U_L^\dagger$, where U_B, U_R, U_T and U_R denote bottom, right, top and left links of the plaquette p_1 (7.8). The disorder operators will rotate only the right link U_R of the plaquette from the left (bottom) end around axis $\hat{\omega}_0$;

$$\begin{aligned}\Sigma_{\vec{\omega}}(p_2,\, p_1)\mathcal{W}_{\alpha\beta}(p_1)\ \Sigma_{\vec{\omega}}^{-1}(p_2,p_1) &= \Big(U_B D^{j=\frac{1}{2}}(\hat{\omega}_0,\omega)U_R U_T^\dagger U_L^\dagger\Big)_{\alpha\beta}\\ &= \Big(U_B D^{j=\frac{1}{2}}(\hat{\omega}_0,\omega)U_B^\dagger U_B U_R U_T^\dagger U_L^\dagger\Big)_{\alpha\beta}\\ &= \Big(U_B D^{j=\frac{1}{2}}(\hat{\omega}_0,\omega)U_B^\dagger \mathcal{W}(p_1)\Big)_{\alpha\beta}\end{aligned}$$

Now we use the following property for Wigner D-function, $U_B\, D^{\frac{1}{2}}(\hat{\omega}_0^{\rm a},\omega)\, U_B^\dagger = D^{\frac{1}{2}}(R^{\rm ab}(U_B^\dagger)\hat{\omega}_0^{\rm b},\, \omega)$ $\equiv D^{\frac{1}{2}}(\hat{\omega'}_0,\, \omega)$ $(\hat{\omega}_0'^{\rm a} \equiv R^{\rm ab}(U_B^\dagger)\hat{\omega}_0^{\rm b})$ which yields

$$\Sigma_{\vec{\omega}}(p_2,p_1)\mathcal{W}_{\alpha\beta}(p_1)\ \Sigma_{\vec{\omega}}^{-1}(p_2,p_1) = \Big(\ D^{j=\frac{1}{2}}(\hat{\omega}_0',\omega)\mathcal{W}(p_1)\Big)_{\alpha\beta}. \tag{E.4}$$

case-II

Consider a plaquette in the middle of the Dirac string and write $\mathcal{W}(s) = \mathcal{W}(x_0+s, y_0) = U_B\, U_R\, U_T^\dagger\, U_L^\dagger$, $0 < s < \mathsf{L}$. Now the disorder operator will rotate the two link $U_L^\dagger$ and U_R from the right and the left around axis $\hat{\omega}_s$ and $\hat{\omega}_{s+1}$ respectively as follows

$$\begin{aligned}\Sigma_{\vec{\omega}}(p_2,p_1)\mathcal{W}_{\alpha\beta}(s)\ \Sigma_{\vec{\omega}}^{-1}(p_2,p_1) &= \Big(U_B D^{j=\frac{1}{2}}(\hat{\omega}_{s+1},\omega)U_R U_T^\dagger U_L^\dagger D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_s,\omega)\Big)_{\alpha\beta}\\ &= \Big(U_B D^{j=\frac{1}{2}}(\hat{\omega}_{s+1},\omega)U_B^\dagger\ U_B U_R U_T^\dagger U_L^\dagger D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_s,\omega)\Big)_{\alpha\beta}\\ &= \Big(U_B D^{j=\frac{1}{2}}(\hat{\omega}_{s+1},\omega)U_B^\dagger\ \mathcal{W}\, D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_s,\omega)\Big)_{\alpha\beta}\end{aligned}$$

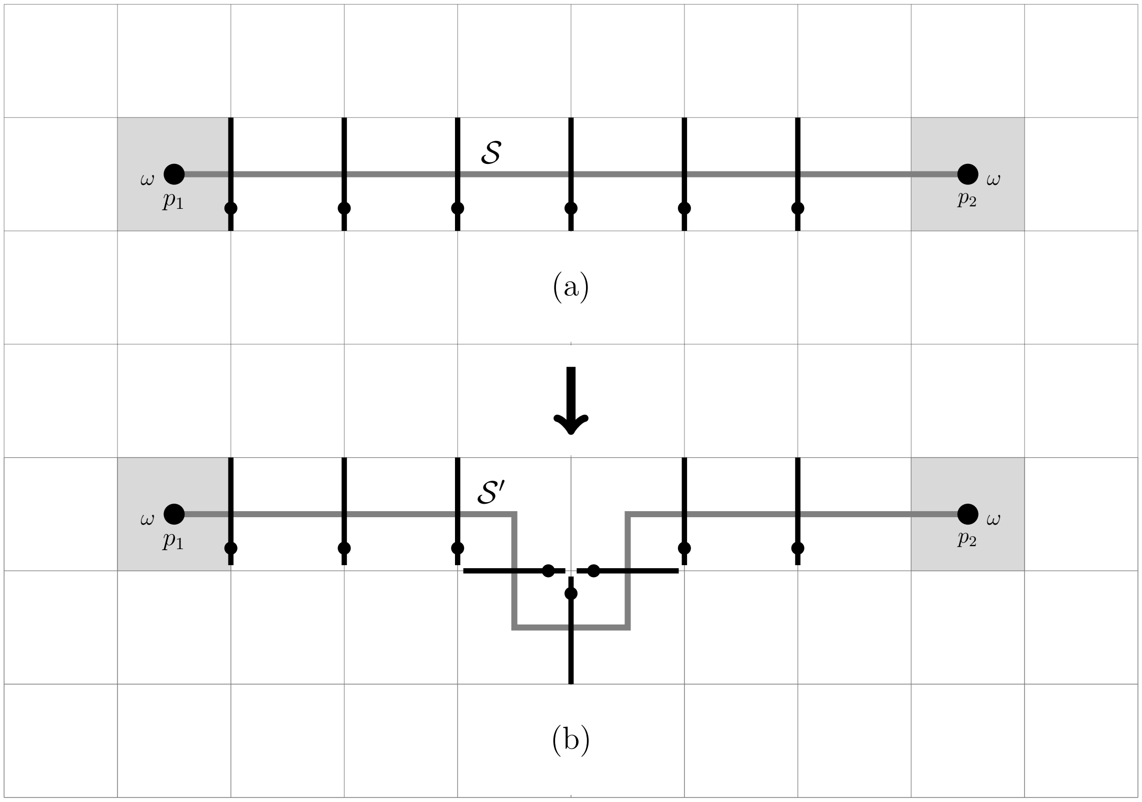


Figure E.1: The deformation of the Dirac string $\mathcal{S}$ by gauge transformations.

Form equation (E.2) we have $\hat{\omega}^{\rm a}_{s+1} = R^{\rm ab}(\mathcal{W}^\dagger(s)U_B)\hat{\omega}_s$;

$$\begin{aligned}\Sigma_{\vec{\omega}}(p_2,p_1)\mathcal{W}_{\alpha\beta}(p)\ \Sigma^{-1}_{\vec{\omega}}(p_2,p_1) &= \Big(U_B U_B^\dagger \mathcal{W}\, D^{j=\frac{1}{2}}(\hat{\omega}_s,\omega)\mathcal{W}^\dagger\, U_B U_B^\dagger \mathcal{W} D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_s,\omega)\Big)_{\alpha\beta}\\ &= \mathcal{W}_{\alpha\beta}(p)\end{aligned}$$

case-III

We now consider plaquette $\mathcal{W}(p_2)$ and write $\mathcal{W}(p_2) = U_B\, U_R\, U_T^\dagger\, U_L^\dagger$. The disorder operators will now rotate only the left link $U_L^\dagger$ of p_2 from the right around axis $\hat{\omega}_{\sf L}$;

$$\begin{aligned}\Sigma_{\vec{\omega}}(p_2,p_1)\mathcal{W}(p_2)\ \Sigma^{-1}_{\vec{\omega}}(p_2,p_0) &= \Big(U_B U_R U_T^\dagger U_L^\dagger D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_{\sf L},\omega)\Big)_{\alpha\beta}\\ &= \Big(\mathcal{W}(p_2) D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_{\sf L},\omega)\Big)_{\alpha\beta}\\ &= \Big(\mathcal{W}(p_2) D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}_{\sf L},\omega)\mathcal{W}^\dagger(p_2)\mathcal{W}(p_2)\Big)_{\alpha\beta}\\ &= \Big(D^{\dagger\, j=\frac{1}{2}}(\hat{\omega}'_{\sf L},\omega)\mathcal{W}(p_2)\Big)_{\alpha\beta}\end{aligned}$$

In above we have used $\hat{\omega}'^{\rm a}_{\sf L} = R^{\rm ab}(\mathcal{W}^\dagger(p_2))\hat{\omega}^{\rm b}_{\sf L}$. We can now easily establish the algebra (7.46) by considering an arbitrary Wilson loop $\mathcal{W}(\mathcal{C})$. If $\mathcal{W}(\mathcal{C})$ encircle one of the two magnetic vortex created by $\Sigma_{\vec{\omega}}(p_2, p_1)$, it will cut the Dirac string $\mathcal{S}$ along a vertical link $U(s, \hat{2}) \in \mathcal{S}$. For such a Wilson loop we can always write $\mathcal{W}(\mathcal{C}) \equiv V_1 U(s; \hat{2}) V_2$ now

$$\begin{aligned}
\Sigma_{\vec{\omega}}(p_2, p_1)\mathcal{W}(\mathcal{C})\Sigma^{-1}_{\vec{\omega}}(p_2, p_1) &= V_1 \Sigma_{\vec{\omega}}(p_2, p_1) U(s, \hat{2}) \Sigma^{-1}_{\vec{\omega}}(p_2, p_1) V_2 \\
&= V_1\, D^{j=\frac{1}{2}}(\hat{\omega}_s, \omega) U(s, \hat{2}) V_2 \\
&= V_1\, D^{j=\frac{1}{2}}(\hat{\omega}_s, \omega) V_1^\dagger\, V_1 U(s, \hat{2}) V_2 \\
&= D^{j=\frac{1}{2}}(\hat{\omega}', \omega)\, \mathcal{W}(\mathcal{C})
\end{aligned}$$

Where axis of rotation $\hat{\omega}'^{\rm a} = R^{\rm ab}(V_1)\hat{\omega}^{\rm b}_s$. Here we note that axis of rotation is also depends upon V_1, therefore is will be different for two magnetic vortex p_1 and p_2. If $\mathcal{W}(\mathcal{C})$ does not enclose any magnetic vortex, it will never cut the Dirac string and therefore we get;

$$\Sigma_{\vec{\omega}}(p_2, p_1)\mathcal{W}(\mathcal{C})\Sigma^{-1}_{\vec{\omega}}(p_2, p_1) = \mathcal{W}(\mathcal{C})$$

BIBLIOGRAPHY

[1] G. 't Hooft, "Gauge theories of the forces between elementary particles," *Scientific American*, vol. 242, no. 6, pp. 104–141, 1980.

[2] K. G. Wilson, "Confinement of quarks," *Physical review D*, vol. 10, no. 8, p. 2445, 1974.

[3] J. B. Kogut, "An introduction to lattice gauge theory and spin systems," *Reviews of Modern Physics*, vol. 51, no. 4, p. 659, 1979.

[4] M. Creutz, *Quarks, gluons and lattices.* Cambridge University Press, 1983, vol. 8.

[5] H. J. Rothe, *Lattice gauge theories: an introduction.* World Scientific Publishing Company, 2012.

[6] C. Gattringer and C. Lang, *Quantum chromodynamics on the lattice: an introductory presentation.* Springer Science & Business Media, 2009, vol. 788.

[7] D. Horn and M. Weinstein, "The t expansion: A nonperturbative analytic tool for Hamiltonian systems," *Physical Review D*, vol. 30, no. 6, p. 1256, 1984.

[8] D. Horn, M. Karliner, and M. Weinstein, "The t expansion and SU(2) lattice gauge theory," *Physical Review D*, vol. 31, no. 10, p. 2589, 1985.

[9] L. C. Hollenberg, "First order analytic diagonalization of lattice QCD," *Physical Review D*, vol. 50, no. 11, p. 6917, 1994.

[10] C. L. Smith and N. Watson, "The shifted coupled cluster method. a new approach to Hamiltonian lattice gauge theories," *Physics Letters B*, vol. 302, no. 4, pp. 463–471, 1993.

[11] D. Schutte, Z. Weihong, and C. Hamer, "Coupled cluster method in Hamiltonian lattice field theory," *Physical Review D*, vol. 55, no. 5, p. 2974, 1997.

[12] R. F. Bishop, N. Ligterink, and N. R. Walet, "Towards a coupled-cluster treatment of SU(n) lattice gauge field theory," *International journal of modern physics B*, vol. 20, no. 30n31, pp. 4992–5007, 2006.

[13] F. W. Group, S. Aoki, Y. Aoki, C. Bernard, T. Blum, G. Colangelo, M. Della Morte, S. Dürr, A. El-Khadra, H. Fukaya *et al.*, "Review of lattice results concerning low-energy particle physics," *The European Physical Journal C*, vol. 74, no. 9, p. 2890, 2014.

[14] S. W. Mages, "Properties of the quark gluon plasma from lattice QCD," Ph.D. dissertation, 2015.

[15] K. Fukushima and T. Hatsuda, "The phase diagram of dense QCD," *Reports on Progress in Physics*, vol. 74, no. 1, p. 014001, 2010.

[16] E. Fradkin, *Field theories of condensed matter physics.* Cambridge University Press, 2013.

[17] A. Altland and B. D. Simons, *Condensed matter field theory.* Cambridge university press, 2010.

[18] M. C. Banuls, K. Cichy, J. I. Cirac, K. Jansen, and S. Kühn, "Tensor networks and their use for lattice gauge theories," *arXiv preprint arXiv:1810.12838*, 2018.

[19] M. Aidelsburger, L. Barbiero, A. Bermudez, T. Chanda, A. Dauphin, D. González-Cuadra, P. R. Grzybowski, S. Hands, F. Jendrzejewski, J. Jünemann *et al.*, "Cold atoms meet lattice gauge theory," *Philosophical Transactions of the Royal Society A*, vol. 380, no. 2216, p. 20210064, 2022.

[20] M. C. Banuls, R. Blatt, J. Catani, A. Celi, J. I. Cirac, M. Dalmonte, L. Fallani, K. Jansen, M. Lewenstein, S. Montangero *et al.*, "Simulating lattice gauge theories within quantum technologies," *The European physical journal D*, vol. 74, no. 8, pp. 1–42, 2020.

[21] U.-J. Wiese, "Ultracold quantum gases and lattice systems: quantum simulation of lattice gauge theories," *Annalen der Physik*, vol. 525, no. 10-11, pp. 777–796, 2013.

[22] F. M. Surace, P. P. Mazza, G. Giudici, A. Lerose, A. Gambassi, and M. Dalmonte, "Lattice gauge theories and string dynamics in Rydberg atom quantum simulators," *Physical Review X*, vol. 10, no. 2, p. 021041, 2020.

[23] N. Klco, M. J. Savage, and J. R. Stryker, "SU(2) non-Abelian gauge field theory in one dimension on digital quantum computers," *Physical Review D*, vol. 101, no. 7, p. 074512, 2020.

[24] C. Kokail, C. Maier, R. van Bijnen, T. Brydges, M. K. Joshi, P. Jurcevic, C. A. Muschik, P. Silvi, R. Blatt, C. F. Roos *et al.*, "Self-verifying variational quantum simulation of lattice models," *Nature*, vol. 569, no. 7756, pp. 355–360, 2019.

[25] E. A. Martinez, C. A. Muschik, P. Schindler, D. Nigg, A. Erhard, M. Heyl, P. Hauke, M. Dalmonte, T. Monz, P. Zoller *et al.*, "Real-time dynamics of lattice gauge theories with a few-qubit quantum computer," *Nature*, vol. 534, no. 7608, pp. 516–519, 2016.

[26] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner *et al.*, "Probing many-body dynamics on a 51-atom quantum simulator," *Nature*, vol. 551, no. 7682, pp. 579–584, 2017.

[27] E. Zohar, J. I. Cirac, and B. Reznik, "Quantum simulations of lattice gauge theories using ultracold atoms in optical lattices," *Reports on Progress in Physics*, vol. 79, no. 1, p. 014401, 2015.

[28] D. B. Kaplan and J. R. Stryker, "Gauss's law, duality, and the Hamiltonian formulation of U(1) lattice gauge theory," *Physical Review D*, vol. 102, no. 9, p. 094515, 2020.

[29] R. Savit, "Duality in field theory and statistical systems," *Reviews of Modern Physics*, vol. 52, no. 2, p. 453, 1980.

[30] E. Fradkin, "Disorder operators and their descendants," *Journal of Statistical Physics*, vol. 167, no. 3-4, pp. 427–461, 2017.

[31] T. Senthil, D. T. Son, C. Wang, and C. Xu, "Duality between (2+ 1) d quantum critical points," *Physics Reports*, vol. 827, pp. 1–48, 2019.

[32] N. Seiberg and S.-H. Shao, "Exotic U(1) symmetries, duality, and fractons in (3+1)-dimensional quantum field theory," *SciPost Physics*, vol. 9, no. 4, p. 046, 2020.

[33] M. P. Fisher, "Duality in low dimensional quantum field theories," *Strong interactions in low dimensions*, pp. 419–438, 2004.

[34] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, "3d dualities from 4d dualities," *Journal of High Energy Physics*, vol. 2013, no. 7, pp. 1–70, 2013.

[35] P. A. M. Dirac, "The theory of magnetic poles," *Physical Review*, vol. 74, no. 7, p. 817, 1948.

[36] H. A. Kramers and G. H. Wannier, "Statistics of the two-dimensional ferromagnet. part i," *Physical Review*, vol. 60, no. 3, p. 252, 1941.

[37] E. Fradkin and L. Susskind, "Order and disorder in gauge systems and magnets," *Physical Review D*, vol. 17, no. 10, p. 2637, 1978.

[38] T. Banks, R. Myerson, and J. Kogut, "Phase transitions in Abelian lattice gauge theories," *Nuclear Physics B*, vol. 129, no. 3, pp. 493–510, 1977.

[39] L. P. Kadanoff and H. Ceva, "Determination of an operator algebra for the two-dimensional ising model," *Physical Review B*, vol. 3, no. 11, p. 3918, 1971.

[40] T. Yoneya, "Z(n) topological excitations in Yang-Mills theories: Duality and confinement," *Nuclear Physics B*, vol. 144, no. 1, pp. 195–218, 1978.

[41] A. M. Polyakov, "Quark confinement and topology of gauge theories," *Nuclear Physics B*, vol. 120, no. 3, pp. 429–458, 1977.

[42] S. Coleman, "Quantum sine-gordon equation as the massive thirring model," *Physical Review D*, vol. 11, no. 8, p. 2088, 1975.

[43] N. Seiberg and E. Witten, "Electric-magnetic duality, monopole condensation, and confinement in n= 2 supersymmetric Yang-Mills theory," *Nuclear Physics B*, vol. 426, no. 1, pp. 19–52, 1994.

[44] N. Seiberg, "Electric-magnetic duality in supersymmetric non-Abelian gauge theories," *Nuclear Physics B*, vol. 435, no. 1-2, pp. 129–146, 1995.

[45] J. Polchinski, "String duality," *Reviews of Modern Physics*, vol. 68, no. 4, p. 1245, 1996.

[46] M. E. Peskin, "Duality in supersymmetric Yang-Mills theory," *arXiv preprint hep-th/9702094*, 1997.

[47] S. Mandelstam, "Quantum electrodynamics without potentials," *Annals of Physics*, vol. 19, no. 1, pp. 1–24, 1962.

[48] S. Mandelstam, "Feynman rules for electromagnetic and Yang-Mills fields from the gauge-independent field-theoretic formalism," *Physical Review*, vol. 175, no. 5, p. 1580, 1968.

[49] S. Mandelstam, "Charge-monopole duality and the phases of non-Abelian gauge theories," *Physical Review D*, vol. 19, no. 8, p. 2391, 1979.

[50] T. T. Wu and C. N. Yang, "Concept of nonintegrable phase factors and global formulation of gauge fields," *Physical Review D*, vol. 12, no. 12, p. 3845, 1975.

[51] A. M. Polyakov, "String representations and hidden symmetries for gauge fields," *Physics Letters B*, vol. 82, no. 2, pp. 247–250, 1979.

[52] A. A. Migdal, "Loop equations and 1n expansion," *Physics Reports*, vol. 102, no. 4, pp. 199–290, 1983.

[53] Y. M. Makeenko and A. A. Migdal, "Exact equation for the loop average in multicolor QCD," *Physics Letters B*, vol. 88, no. 1-2, pp. 135–137, 1979.

[54] Y. M. Makeenko and A. Migdal, "Quantum chromodynamics as dynamics of loops," *Nuclear Physics B*, vol. 188, no. 2, pp. 269–316, 1981.

[55] F. Gliozzi, T. Regge, and M. Virasoro, "Gauge fields and phonon excitations in a condensate of dual strings," *Phys. Lett., B;(Netherlands)*, vol. 81, no. 2, 1979.

[56] J. Kogut and L. Susskind, "Hamiltonian formulation of Wilson's lattice gauge theories," *Physical Review D*, vol. 11, no. 2, p. 395, 1975.

[57] R. Giles, "Reconstruction of gauge potentials from Wilson loops," *Physical Review D*, vol. 24, no. 8, p. 2160, 1981.

[58] C. Rovelli, *Quantum gravity.* Cambridge university press, 2004.

[59] A. Ashtekar, "New variables for classical and quantum gravity," *Physical review letters*, vol. 57, no. 18, p. 2244, 1986.

[60] R. Gambini and J. Pullin, *Loops, knots, gauge theories and quantum gravity.* Cambridge University Press, 2000.

[61] M. Mathur, "Harmonic oscillator pre-potentials in SU(2) lattice gauge theory," *Journal of Physics A: Mathematical and General*, vol. 38, no. 46, p. 10015, 2005.

[62] M. Mathur, "Loop states in lattice gauge theory," *Physics Letters B*, vol. 640, no. 5-6, pp. 292–296, 2006.

[63] H. Sharatchandra, "Local observables in non-Abelian gauge theories," *Nuclear Physics B*, vol. 196, no. 1, pp. 62–82, 1982.

[64] B. Gnanapragasam and H. Sharatchandra, "Abelianization of non-Abelian lattice gauge theories," *Physical Review D*, vol. 45, no. 4, p. R1010, 1992.

[65] R. Anishetty, "Local dynamics on a gauge-invariant basis of non-Abelian gauge theories," *Physical Review D*, vol. 44, no. 6, p. 1895, 1991.

[66] F. Gliozzi and M. Virasoro, "The interaction among dual strings as a manifestation of the gauge group," *Nuclear Physics B*, vol. 164, pp. 141–151, 1980.

[67] D. Robson and D. Webber, "Gauge covariance in lattice field theories," *Zeitschrift für Physik C Particles and Fields*, vol. 15, no. 3, pp. 199–226, 1982.

[68] P. Majumdar and H. Sharatchandra, "Duality transformation for (3+1)-dimensional Yang-Mills theory," *International Journal of Modern Physics A*, vol. 17, no. 02, pp. 175–186, 2002.

[69] M. Halpern, "Field-strength and dual variable formulations of gauge theory," *Physical Review D*, vol. 19, no. 2, p. 517, 1979.

[70] J. Goldstone and R. Jackiw, "Unconstrained temporal gauge for Yang-Mills theory," *Physics Letters B*, vol. 74, no. 1-2, pp. 81–84, 1978.

[71] W. Furmanski and A. Kolawa, "Yang-Mills vacuum: An attempt at lattice loop calculus," *Nuclear Physics B*, vol. 291, pp. 594–628, 1987.

[72] R. Anishetty and H. Sharatchandra, "Duality transformation for non-Abelian lattice gauge theories," *Physical review letters*, vol. 65, no. 7, p. 813, 1990.

[73] G. Burgio, R. De Pietri, H. Morales-Tecotl, L. Urrutia, and J. Vergara, "The basis of the physical Hilbert space of lattice gauge theories," *Nuclear Physics B*, vol. 566, no. 3, pp. 547–561, 2000.

[74] N. Ligterink, N. Walet, and R. Bishop, "A many-body treatment of Hamiltonian lattice gauge theory," *Nuclear Physics A*, vol. 663, pp. 983c–986c, 2000.

[75] V. Müller and W. Rühl, "The energy gap of SU(2) lattice gauge theory in (2+1) dimensions," *Nuclear Physics B*, vol. 230, no. 1, pp. 49–61, 1984.

[76] J. B. Bronzan, "Explicit Hamiltonian for SU(2) lattice gauge theory," *Physical Review D*, vol. 31, no. 8, p. 2020, 1985.

[77] J. B. Bronzan and T. E. Vaughan, "A gauge-fixed Hamiltonian for lattice QCD," *Physical Review D*, vol. 47, no. 8, p. 3543, 1993.

[78] M. Mathur and T. Sreeraj, "Lattice gauge theories and spin models," *Physical Review D*, vol. 94, no. 8, p. 085029, 2016.

[79] M. Mathur and T. P. Sreeraj, "Canonical transformations and loop formulation of SU(N) lattice gauge theories," *Physical Review D*, vol. 92, no. 12, p. 125018, 2015.

[80] F. Lunev, "Three dimensional Yang-Mills theory in gauge invariant variables," *Physics Letters B*, vol. 295, no. 1-2, pp. 99–103, 1992.

[81] M. Bauer, D. Z. Freedman, and P. E. Haagensen, "Spatial geometry of the electric field representation of non-Abelian gauge theories," *Nuclear Physics B*, vol. 428, no. 1-2, pp. 147–168, 1994.

[82] P. E. Haagensen and K. Johnson, "Yang-Mills fields and Riemannian geometry," *Nuclear Physics B*, vol. 439, no. 3, pp. 597–616, 1995.

[83] P. E. Haagensen, K. Johnson, and C. Lam, "Gauge invariant geometric variables for Yang-Mills theory," *Nuclear Physics B*, vol. 477, no. 1, pp. 273–292, 1996.

[84] P. Majumdar, "Duality transformation of non-Abelian gauge theories," 2009.

[85] I. Mitra and H. Sharatchandra, "Dreibein as prepotential for three-dimensional Yang-Mills theory," *Advances in High Energy Physics*, vol. 2017, 2017.

[86] D. Karabali and V. Nair, "A gauge-invariant Hamiltonian analysis for non-Abelian gauge theories in (2+1) dimensions," *Nuclear Physics B*, vol. 464, no. 1-2, pp. 135–152, 1996.

[87] V. Nair and A. Yelnikov, "On the invariant measure for the Yang-Mills configuration space in (3+1) dimensions," *Nuclear Physics B*, vol. 691, no. 1-2, pp. 182–194, 2004.

[88] L. Freidel, R. G. Leigh, and D. Minic, "Towards a solution of pure Yang-Mills theory in (3+1) dimensions," *Physics Letters B*, vol. 641, no. 1, pp. 105–111, 2006.

[89] R. Gambini, L. Leal, and A. Trias, "Loop calculus for lattice gauge theories," *Physical Review D*, vol. 39, no. 10, p. 3127, 1989.

[90] M. Mathur and T. Sreeraj, "From lattice gauge theories to hydrogen atoms," *Physics Letters B*, vol. 749, pp. 137–143, 2015.

[91] M. Mathur and A. Rathor, "Exact duality and local dynamics in SU(n) lattice gauge theory," *Physical Review D*, vol. 107, no. 7, p. 074504, 2023.

[92] C. W. B. Davoudi, A. Balantekin, T. Bhattacharya, M. Carena, W. A. de Jong, P. Draper, A. El-Khadra, N. Gemelke, M. Hanada, D. Kharzeev *et al.*, "Quantum simulation for high energy physics," *arXiv preprint arXiv:2204.03381*, 2022.

[93] F. J. Wegner, "Duality in generalized ising models and phase transitions without local order parameters," *Journal of Mathematical Physics*, vol. 12, no. 10, pp. 2259–2272, 1971.

[94] A. Ukawa, P. Windey, and A. H. Guth, "Dual variables for lattice gauge theories and the phase structure of z (n) systems," *Physical Review D*, vol. 21, no. 4, p. 1013, 1980.

[95] G. Hooft, "On the phase transition towards permanent quark confinement," *Nuclear Physics: B*, vol. 138, no. 1, pp. 1–25, 1978.

[96] G. Mack and V. Petkova, "Comparison of lattice gauge theories with gauge groups Z_2 and SU(2)," *Annals of Physics*, vol. 123, no. 2, pp. 442–467, 1979.

[97] E. Tomboulis, "'t Hooft loop in SU(2) lattice gauge theories," *Physical Review D*, vol. 23, no. 10, p. 2371, 1981.

[98] D. Horn, M. Weinstein, and S. Yankielowicz, "Hamiltonian approach to Z(N) lattice gauge theories," *Physical Review D*, vol. 19, no. 12, p. 3715, 1979.

[99] R. P. Feynman, "Simulating physics with computers," in *Feynman and computation*. CRC Press, 2018, pp. 133–153.

[100] S. Lloyd, "Universal quantum simulators," *Science*, vol. 273, no. 5278, pp. 1073–1078, 1996.

[101] A. Y. Kitaev, "Fault-tolerant quantum computation by anyons," *Annals of Physics*, vol. 303, no. 1, pp. 2–30, 2003.

[102] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, "Non-Abelian anyons and topological quantum computation," *Reviews of Modern Physics*, vol. 80, no. 3, p. 1083, 2008.

[103] J. Preskill, "Lecture notes for physics 219: Quantum computation," *Caltech Lecture Notes*, p. 7, 1999.

[104] S. Sachdev, "Z2 gauge theory."

[105] A. J. Leggett, "The kitaev models," 2013.

[106] M. Mathur and A. Rathor, "SU(N) toric code and non-Abelian anyons," *Physical Review A*, vol. 105, no. 5, p. 052423, 2022.

[107] M. D. Schulz, S. Dusuel, R. Orus, J. Vidal, and K. P. Schmidt, "Breakdown of a perturbed topological phase," *New Journal of Physics*, vol. 14, no. 2, p. 025005, 2012.

[108] J. R. Wootton, V. Lahtinen, B. Doucot, and J. K. Pachos, "Engineering complex topological memories from simple Abelian models," *Annals of Physics*, vol. 326, no. 9, pp. 2307–2314, 2011.

[109] F. Bais and J. Slingerland, "Condensate-induced transitions between topologically ordered phases," *Physical Review B*, vol. 79, no. 4, p. 045316, 2009.

[110] M. A. Levin and X.-G. Wen, "String-net condensation: A physical mechanism for topological phases," *Physical Review B*, vol. 71, no. 4, p. 045110, 2005.

[111] D. Horn, "Phase structure of the Z(2) gauge and matter theory," *Physics Reports*, vol. 67, no. 1, pp. 103–107, 1980.

[112] C. K. Altes, “Duality for Z(N) gauge theories,” *Nuclear Physics B*, vol. 142, no. 3, pp. 315–326, 1978.

[113] E. Fradkin and S. Raby, “Real-space renormalization-group scheme for spin and gauge systems,” *Physical Review D*, vol. 20, no. 10, p. 2566, 1979.

[114] C. Itzykson, M. E. Peskin, and J. Zuber, “Roughening of Wilson’s surface,” *Physics Letters B*, vol. 95, no. 2, pp. 259–264, 1980.

[115] M. E. Peskin, “Mandelstam-’t Hooft duality in Abelian lattice models,” *Annals of Physics*, vol. 113, no. 1, pp. 122–152, 1978.

[116] G. Hooft, “A property of electric and magnetic flux in non-Abelian gauge theories,” *Nuclear physics: B*, vol. 153, pp. 141–160, 1979.

[117] C. Itzykson, A. Jaffe, H. Lehmann, P. Mitter, I. Singer, R. Stora *et al.*, *Recent developments in gauge theories.* Springer, 1980.

[118] T. D. Schultz, D. C. Mattis, and E. H. Lieb, “Two-dimensional ising model as a soluble problem of many fermions,” *Reviews of Modern Physics*, vol. 36, no. 3, p. 856, 1964.

[119] E. Fradkin, M. Srednicki, and L. Susskind, “Fermion representation for the Z_2 lattice gauge theory in 2+ 1 dimensions,” *Physical Review D*, vol. 21, no. 10, p. 2885, 1980.

[120] M. Srednicki, “Hidden fermions in Z(2) theories,” *Physical Review D*, vol. 21, no. 10, p. 2878, 1980.

[121] T. Sugihara, “Matrix product representation of gauge invariant states in a Z_2 lattice gauge theory,” *Journal of High Energy Physics*, vol. 2005, no. 07, p. 022, 2005.

[122] B. Swingle and X.-G. Wen, “Topological properties of tensor network states from their local gauge and local symmetry structures,” *arXiv preprint arXiv:1001.4517*, 2010.

[123] S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba, and H. Tasaki, “On the definition of entanglement entropy in lattice gauge theories,” *Journal of High Energy Physics*, vol. 2015, no. 6, pp. 1–29, 2015.

[124] E. Zohar and B. Reznik, “Confinement and lattice quantum-electrodynamic electric flux tubes simulated with ultracold atoms,” *Physical review letters*, vol. 107, no. 27, p. 275301, 2011.

[125] P. Buividovich and M. Polikarpov, “Entanglement entropy in gauge theories and the holographic principle for electric strings,” *Physics Letters B*, vol. 670, no. 2, pp. 141–145, 2008.

[126] F. Wegner, "Duality in generalized ising models," *Topol. Asp. Condens. Matter Phys.(Oxford University Press, 2017) pp*, pp. 219–240, 2015.

[127] G. H. Wannier, "The statistical problem in cooperative phenomena," *Reviews of Modern Physics*, vol. 17, no. 1, p. 50, 1945.

[128] J. B. Kogut, "The lattice gauge theory approach to quantum chromodynamics," *Reviews of Modern Physics*, vol. 55, no. 3, p. 775, 1983.

[129] R. Loll, "Independent SU(2)-loop variables and the reduced configuration space of su (2)-lattice gauge theory," *Nuclear Physics B*, vol. 368, no. 1, pp. 121–142, 1992.

[130] N. Watson, "Solution of the SU(2) Mandelstam constraints," *Physics Letters B*, vol. 323, no. 3-4, pp. 385–392, 1994.

[131] N. J. Watson, "Gauge-invariant variables and Mandelstam constraints in SU(2) gauge theory," *Nuclear Physics B-Proceedings Supplements*, vol. 39, no. 2-3, pp. 224–227, 1995.

[132] A. M. Polyakov, "Gauge fields as rings of glue," *Nuclear Physics B*, vol. 164, pp. 171–188, 1980.

[133] A. Jevicki and B. Sakita, "Loop-space representation and the large-n behavior of the one-plaquette Kogut-Susskind Hamiltonian," *Physical Review D*, vol. 22, no. 2, p. 467, 1980.

[134] B. Brügmann, "Method of loops applied to lattice gauge theory," *Physical Review D*, vol. 43, no. 2, p. 566, 1991.

[135] C. Di Bartolo, R. Gambini, and L. Leal, "Hamiltonian lattice gauge theories in a loop-dependent magnetic representation," *Physical Review D*, vol. 39, no. 6, p. 1756, 1989.

[136] R. Loll, "Yang-Mills theory without Mandelstam constraints," *Nuclear Physics B*, vol. 400, no. 1-3, pp. 126–144, 1993.

[137] M. Mathur, "Loop approach to lattice gauge theories," *Nuclear Physics B*, vol. 779, no. 1-2, pp. 32–62, 2007.

[138] R. Anishetty, M. Mathur, and I. Raychowdhury, "Irreducible SU(3) Schwinger Bosons," *Journal of mathematical physics*, vol. 50, no. 5, p. 053503, 2009.

[139] A. Ramesh, M. Mathur, and R. Indrakshi, "Prepotential formulation of SU(3) lattice gauge theory," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 3, p. 035403, 2009.

[140] M. Mathur, I. Raychowdhury, and R. Anishetty, "SU(n) irreducible Schwinger Bosons," *Journal of mathematical physics*, vol. 51, no. 9, p. 093504, 2010.

[141] M. Mathur, I. Raychowdhury, and T. Sreeraj, "Invariants, projection operators and SU(N)× SU(N) irreducible Schwinger Bosons," *Journal of mathematical physics*, vol. 52, no. 11, p. 113505, 2011.

[142] R. Anishetty and I. Raychowdhury, "SU(2) lattice gauge theory: Local dynamics on nonintersecting electric flux loops," *Physical Review D*, vol. 90, no. 11, p. 114503, 2014.

[143] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, "Quantum electrodynamics on a lattice: A Hamiltonian variational approach to the physics of the weak-coupling region," *Physical Review D*, vol. 19, no. 2, p. 619, 1979.

[144] Z. Davoudi, A. F. Shaw, and J. R. Stryker, "General quantum algorithms for Hamiltonian simulation with applications to a non-Abelian lattice gauge theory," *arXiv preprint arXiv:2212.14030*, 2022.

[145] A. Salmela, "An algebraic method for solving the SU(3) Gauss law," *Journal of Mathematical Physics*, vol. 44, no. 6, pp. 2521–2533, 2003.

[146] H. Reinhardt, "Resolution of Gauss's law in Yang-Mills theory by gauge-invariant projection: Topology and magnetic monopoles," *Nuclear Physics B*, vol. 503, no. 1-2, pp. 505–529, 1997.

[147] L. Lootens, C. Delcamp, and F. Verstraete, "Dualities in one-dimensional quantum lattice models: topological sectors," *arXiv preprint arXiv:2211.03777*, 2022.

[148] R. Oeckl and H. Pfeiffer, "The dual of pure non-Abelian lattice gauge theory as a spin foam model," *Nuclear Physics B*, vol. 598, no. 1-2, pp. 400–426, 2001.

[149] R. Anishetty, P. Majumdar, and H. Sharatchandra, "Dual gluons and monopoles in 2+1 dimensional Yang-Mills theory," *Physics Letters B*, vol. 478, no. 1-3, pp. 373–378, 2000.

[150] V. Kasper, T. V. Zache, F. Jendrzejewski, M. Lewenstein, and E. Zohar, "Non-Abelian gauge invariance from dynamical decoupling," *Physical Review D*, vol. 107, no. 1, p. 014506, 2023.

[151] E. Zohar, J. I. Cirac, and B. Reznik, "Cold-atom quantum simulator for SU(2) Yang-Mills lattice gauge theory," *Physical review letters*, vol. 110, no. 12, p. 125304, 2013.

[152] J. R. Stryker, "Oracles for Gauss's law on digital quantum computers," *Physical Review A*, vol. 99, no. 4, p. 042301, 2019.

[153] S. Ashkenazi and E. Zohar, "Duality as a feasible physical transformation for quantum simulation," *Physical Review A*, vol. 105, no. 2, p. 022431, 2022.

[154] T. G. Kovacs and E. T. Tomboulis, "Vortices and confinement," *NATO Sci. Ser. C*, vol. 553, pp. 315–326, 2000.

[155] T. G. Kovacs and E. T. Tomboulis, "Vortex structure of the vacuum and confinement," *Nucl. Phys. B Proc. Suppl.*, vol. 94, pp. 518–521, 2001.

[156] L. G. Yaffe, "Confinement in SU(N) lattice gauge theories," *Physical Review D*, vol. 21, no. 6, p. 1574, 1980.

[157] H. Reinhardt, "On 't Hooft's loop operator," *Physics Letters B*, vol. 557, no. 3-4, pp. 317–323, 2003.

[158] J.-M. Drouffe and J.-B. Zuber, "Strong coupling and mean field methods in lattice gauge theories," *Physics Reports*, vol. 102, no. 1-2, pp. 1–119, 1983.

[159] T. G. Kovacs and E. T. Tomboulis, "Computation of the vortex free energy in SU(2) gauge theory," *Physical Review Letters*, vol. 85, no. 4, p. 704, 2000.

[160] P. De Forcrand and D. Noth, "Precision lattice calculation of SU(2) 't Hooft loops," *Physical Review D*, vol. 72, no. 11, p. 114501, 2005.

[161] H. Reinhardt and D. Epple, "The't hooft loop in the hamiltonian approach to yang-mills theory in coulomb gauge," *Physical Review D*, vol. 76, no. 6, p. 065015, 2007.

[162] J. Greensite, *An introduction to the confinement problem.* Springer, 2011, vol. 821.

[163] J. Greensite, "The confinement problem in lattice gauge theory," *Progress in Particle and Nuclear Physics*, vol. 51, no. 1, pp. 1–83, 2003.

[164] L. Del Debbio, M. Faber, J. Greensite, and Š. Olejník, "Center dominance and z 2 vortices in su (2) lattice gauge theory," *Physical Review D*, vol. 55, no. 4, p. 2298, 1997.

[165] K. Langfeld, H. Reinhardt, and O. Tennert, "Confinement and scaling of the vortex vacuum of su (2) lattice gauge theory," *Physics Letters B*, vol. 419, no. 1-4, pp. 317–321, 1998.

[166] K. Mallesh and N. Mukunda, "The algebra and geometry of SU(3) matrices," *Pramana*, vol. 49, pp. 371–383, 1997.

[167] H. Georgi, *Lie algebras in particle physics: from isospin to unified theories.* Taylor & Francis, 2000.

[168] G. Mack and V. Petkova, "Z 2 monopoles in the standard su (2) lattice gauge theory model," *Zeitschrift für Physik C Particles and Fields*, vol. 12, pp. 177–184, 1982.

[169] G. Mack and E. Pietarinen, "Monopoles, vortices and confinement," *Nuclear Physics B*, vol. 205, no. 2, pp. 141–167, 1982.

[170] L. Tagliacozzo, A. Celi, A. Zamora, and M. Lewenstein, "Optical Abelian lattice gauge theories," *Annals of Physics*, vol. 330, pp. 160–191, 2013.

[171] Z.-Y. Zhou, G.-X. Su, J. C. Halimeh, R. Ott, H. Sun, P. Hauke, B. Yang, Z.-S. Yuan, J. Berges, and J.-W. Pan, "Thermalization dynamics of a gauge theory on a quantum simulator," *Science*, vol. 377, no. 6603, pp. 311–314, 2022.

[172] L. Homeier, C. Schweizer, M. Aidelsburger, A. Fedorov, and F. Grusdt, "Z_2 lattice gauge theories and Kitaev's toric code: A scheme for analog quantum simulation," *Physical Review B*, vol. 104, no. 8, p. 085138, 2021.

[173] Y. Tong, V. V. Albert, J. R. McClean, J. Preskill, and Y. Su, "Provably accurate simulation of gauge theories and Bosonic systems," *arXiv preprint arXiv:2110.06942*, 2021.

[174] Z. Davoudi, M. Hafezi, C. Monroe, G. Pagano, A. Seif, and A. Shaw, "Towards analog quantum simulations of lattice gauge theories with trapped ions," *Physical Review Research*, vol. 2, no. 2, p. 023015, 2020.

[175] H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, and H. P. Büchler, "A Rydberg quantum simulator," *Nature Physics*, vol. 6, no. 5, pp. 382–388, 2010.

[176] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum theory of angular momentum*. World Scientific, 1988.